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On the effects of nonlinear boundary conditions in diffusive logistic equations on bounded domains ☆

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Abstract

We study a diffusive logistic equation with nonlinear boundary conditions. The equation arises as a model for a population that grows logistically inside a patch and crosses the patch boundary at a rate that depends on the population density. Specifically, the rate at which the population crosses the boundary is assumed to decrease as the density of the population increases. The model is motivated by empirical work on the Glanville fritillary butterfly. We derive local and global bifurcation results which show that the model can have multiple equilibria and in some parameter ranges can support Allee effects. The analysis leads to eigenvalue problems with nonstandard boundary conditions.

Keywords: Reaction–diffusion; Logistic equation; Nonlinear boundary conditions; Allee effect; Eigenvalue problems; Bifurcation

1. Introduction and background

In this article we continue the examination of the diffusive logistic problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= d\nabla^2 u + ru(1-u) && \text{in } \Omega \times (0,\infty), \\ \alpha(u)\nabla u \cdot \vec{\eta} + \big(1-\alpha(u)\big)u &= 0 && \text{on } \partial\Omega \times (0,\infty), \end{aligned} \tag{1.1}$$

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which we began in [5]. In [5], in (1.1), $u = u(x,t)$ designates the density of a biological species at spatial location x and time t, and Ω designates a focal patch of habitat for the species. Mathematically Ω is a bounded open domain in $\mathbb{R}^N$ (in applications N is usually 1, 2, or 3) with sufficiently smooth boundary, d and r are positive parameters giving the diffusion rate and intrinsic growth rate for the species, respectively, and $\alpha(\cdot)$ is a smooth function of the density u which describes the behavior of the species along the boundary. The carrying capacity in (1.1) has been scaled to equal 1.

Our interest in (1.1) has been motivated by some empirical work by the ecologist Ilkka Hanski and his collaborators [10] on mechanisms producing an Allee effect in the Glanville fritillary butterfly. A biological species is said to exhibit an Allee effect in a habitat patch if its *per capita* rate of growth in the patch is *increasing* at low densities. If the growth rate in the patch is actually negative when the density is below some threshold value, the species is said to exhibit a strong Allee effect. In such an event, the species could not be expected to establish itself in the patch if introduced into the patch at a low enough density. In other words, it could not invade the patch. If the *per capita* growth rate remains positive at low densities, the species is said to exhibit a weak Allee effect. In this case, the species can be expected to invade the patch, *if* the patch is sufficiently large. However, the minimal patch size for invasion is larger for a species exhibiting a weak Allee effect than for another species with the same maximal *per capita* growth rate if the other species' *per capita* growth rate is a decreasing function of population density, as is the case with logistic growth.

In [10], Kuussaari et al. demonstrate empirically that having emigration rates from butterfly habitat patches increase as butterfly population density near the edge of habitat patches decreases is a possible mechanism for inducing an Allee effect in the Glanville fritillary butterfly. In [5], we propose (1.1) as a mathematical model capturing this mechanism. To this end, we require that $\alpha(u)$ be nondecreasing in u and that

$$\alpha\big([0,1]\big) \subseteq [0,1]. \tag{1.2}$$

Since the local carrying capacity of the species inside Ω is normalized without loss of generality to the value 1, $u \in [0,1]$ represents a density less than or equal to the local carrying capacity. When $\alpha(u) \in [0,1]$, the boundary condition

$$\alpha(u)\nabla u \cdot \vec{\eta} + \big(1-\alpha(u)\big)u = 0 \tag{1.3}$$

represents a tension between a tendency for the species to be lost through the boundary of the patch to the environs surrounding the patch and a tendency for the species to remain in the patch. As the value of $\alpha(u)$ increases the tendency of the species to remain in the patch becomes more dominant. (If $\alpha(u) = 0$, (1.3) is the Dirichlet condition

$$u = 0,$$

while if $\alpha(u) = 1$, it is the Neumann condition

$$\nabla u \cdot \vec{\eta} = 0.)$$

Consequently, as u increases from 0 to 1, the tendency of the species to emigrate from Ω is nonincreasing, and is strictly decreasing whenever α is strictly increasing, capturing the mechanism in [10]. In the special case that $\alpha(u) \equiv \alpha^* \in [0, 1]$, the dynamics of (1.1) are well known [6]. Depending only on the ratio r/d, either all nonnegative solutions to (1.1) tend to 0 over time or all nonnegative nontrivial solutions to (1.1) tend to a unique equilibrium which is positive throughout Ω. As a result, an Allee effect is not possible when $\alpha(u) \equiv \alpha^* \in [0, 1]$.

In [5], we began our examination of (1.1) and showed that an Allee effect can be expected for appropriate values of the ratio r/d so long as $\alpha(u)$ is sufficiently small for u near 0 and sufficiently large for u less than but near 1. We obtained our results in [5] via a linearized stability analysis and comparison principles based on upper and lower solution techniques. From the standpoint of exhibiting a model that predicts an Allee effect on the basis of the mechanism delineated by Hanski and his collaborators, the analysis in [5] is quite satisfactory. However, the observations in [5] represent only a first step toward a thorough understanding of the asymptotic behavior of positive solutions to (1.1).

A more detailed analysis of (1.1) leads to a number of intriguing mathematical challenges and provides enhanced insight into how Allee effects arise in the model outcomes. The asymptotic disposition of (1.1) depends strongly on what specific assumptions are placed upon the density-dependent term $\alpha(u)$, beyond requiring that $\alpha(u)$ be nondecreasing in u and that (1.2) hold. Indeed, there appears to be a range of possibilities that is beyond the scope of a single article. Consequently, our approach here will be to identify what we view as the most significant additional factors in determining the asymptotic behavior of solutions to (1.1) and then to treat some interesting and important cases. We plan to examine further cases in subsequent work.

The most important additional assumption that one places upon $\alpha(u)$ regards the value of $\alpha(0)$. If $\alpha(0) > 0$, we may assume without loss of generality that

$$\alpha(\mathbb{R}) \subseteq (\delta, R) \tag{1.4}$$

for some $0 < \delta < R < \infty$. In this case, the results of Amann [2] may be employed to conclude that (1.1) is well-posed. It follows from the logistic form of the reaction term in (1.1) that positive solutions to (1.1) exist and are bounded for all positive time.

If $\alpha(0) = 0$, then (1.3) can be expressed as

$$u\big(\beta(u)\nabla u \cdot \vec{\eta} + \big(1 - \alpha(u)\big)\big) = 0 \tag{1.5}$$

on $\partial\Omega \times (0, \infty)$, where $\beta(u)$ in (1.5) is given by

$$\beta(u) = \begin{cases} \frac{\alpha(u)}{u}, & u \neq 0, \\ \alpha'(0), & u = 0. \end{cases}$$

As a result, if $\alpha(0) = 0$, the boundary condition in (1.1) can be satisfied in more than one way and hence (1.1) is not well-posed in this case. One possibility is $u \equiv 0$ on $\partial\Omega$. However, if the initial data for (1.1) is positive throughout $\bar{\Omega}$, a corresponding solution u could not be zero on $\partial\Omega$ for $t \in (0, t_0)$ for some $t_0 > 0$. In this event, such a u must satisfy

$$\beta(u)\nabla u \cdot \vec{\eta} + \big(1 - \alpha(u)\big) = 0 \tag{1.6}$$

for $x \in \partial\Omega$ and $t \in (0, t_0)$. Since $\alpha(u)$ is nondecreasing, the definition of β guarantees that $\beta(u) \geqslant 0$ for $u \geqslant 0$. If we make the additional assumption that

$$\alpha'(0) > 0, \tag{1.7}$$

then $\beta(u) > 0$ for $u \geqslant 0$ and we may assume that $\beta(u)$ satisfies (1.4) for appropriate values of δ and R. So the well-posedness of

$$\begin{aligned} &\frac{\partial u}{\partial t} = d\nabla^2 u + ru(1-u) && \text{in } \Omega \times (0,\infty), \\ &\beta(u)\nabla u \cdot \vec{\eta} + \big(1-\alpha(u)\big) = 0 && \text{on } \partial\Omega \times (0,\infty) \end{aligned} \tag{1.8}$$

follows from [2], as was the case with (1.1) when $\alpha(0) > 0$. Strong maximum principle arguments and the logistic form of the reaction-term guarantee that a solution u of (1.8) with $u(x,0) > 0$ on $\bar{\Omega}$ exists for all positive time and satisfies

$$0 < u(x,t) < M$$

for $x \in \bar{\Omega}$ and $t \geqslant 0$, where $M = M(u(x,0))$. Consequently, if (1.7) holds, a solution to (1.1) with initial data that is positive on $\bar{\Omega}$ is uniquely determined as the solution to (1.8).

In our preceding paper [5] we treated (1.1) in the case in which

$$\alpha(u) = \alpha_0 \quad \text{for } u \leqslant u_1 \quad \text{and} \quad \alpha(u) = 1 \quad \text{for } u \geqslant u_2, \tag{1.9}$$

where $0 < u_1 < u_2 < 1$. Consequently, in light of the preceding discussion, we will restrict our attention in this article to cases in which $\alpha(0) > 0$ or $\alpha(0) = 0$ but $\alpha'(0) > 0$.

The asymptotics of (1.1) also depend significantly upon the value of $\alpha(1)$. When $\alpha(1) < 1$, $u \equiv 0$ is the only spatially homogeneous equilibrium for (1.1). Moreover, we may assume without loss of generality that for an appropriate value of $R \in (0,1)$, $\alpha(u) < R$ for $u \geqslant 0$. On the other hand, if $\alpha(1) = 1$, (1.1) admits both $u \equiv 0$ and $u \equiv 1$ as equilibria. Given our examination in [5] of (1.1) when $\alpha(u)$ satisfies (1.9), we will assume in this article that

$$\alpha'(1) > 0 \tag{1.10}$$

whenever $\alpha(1) = 1$.

As previously noted, we showed in [5] that for appropriate choices of $\alpha(u)$ and the ratio r/d, a species density described by (1.1) exhibits an Allee effect in the habitat Ω. More specifically, we showed that solutions to (1.1) with initial data above a certain threshold remain above that threshold for all subsequent time, while solutions to (1.1) with initial data below a second smaller threshold tend over time to 0. In order to demonstrate that such is the case, we compared solutions to (1.1) for the $\alpha(u)$ in question to solutions to (1.1) with $\alpha(u) \equiv \alpha^* \in [0,1]$. As we have noted, the dynamics of (1.1) when $\alpha(u) \equiv \alpha^*$ are well-understood [6]. Namely, when the average growth rate over Ω of the species in question at low densities is positive, positive solutions to (1.1) tend over time to a globally attracting equilibrium which is positive in Ω. Otherwise, positive solutions to (1.1) tend to zero. The average growth rate $\sigma = \sigma(\alpha^*, r, d)$ at low densities is given by the principal eigenvalue in the elliptic problem

$$\begin{aligned} &d\nabla^2\phi + r\phi = \sigma\phi && \text{in } \Omega, \\ &\alpha^*\nabla\phi \cdot \vec{\eta} + (1-\alpha^*)\phi = 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.11}$$

The average growth rate σ in (1.11) is related to the demographic, geometric and habitat interface parameters in the model (1.1) via the formula

$$\sigma = \sigma(\alpha^*, r, d) = r - d\lambda^1_{\alpha^*}(\Omega), \tag{1.12}$$

where $\lambda = \lambda^1_{\alpha^*}(\Omega)$ is the principal nonnegative eigenvalue of $-\nabla^2$ on Ω relative to the given boundary condition; i.e., the principal eigenvalue in

$$\begin{aligned} -\nabla^2\phi &= \lambda\phi && \text{in } \Omega, \\ \alpha^*\nabla\phi\cdot\vec{\eta} + (1-\alpha^*)\phi &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.13}$$

When $\alpha^* = 1$, $\lambda^1_{\alpha^*}(\Omega)$ in (1.13) is zero. Otherwise, $\lambda^1_{\alpha^*}(\Omega) > 0$. Indeed, $\lambda^1_{\alpha^*}(\Omega)$ is a smooth strictly decreasing function of $\alpha^* \in [0,1]$ and it follows immediately from (1.12) that

$$\sigma(\alpha^*, r, d) > 0 \quad \Leftrightarrow \quad r/d > \lambda^1_{\alpha^*}(\Omega). \tag{1.14}$$

If we now let $\phi_{\alpha^*} = \phi_{\alpha^*}(r,d)$ denote the unique positive eigenfunction in (1.11) satisfying

$$\int_\Omega \phi^2_{\alpha^*}\,dx = 1$$

and $u_{\alpha^*} = u_{\alpha^*}(r,d)$ denote the globally attracting positive equilibrium for (1.1) with $\alpha(u) \equiv \alpha^*$ (which exists when $r/d > \lambda^1_{\alpha^*}(\Omega)$), we can readily summarize the main result of [5]. To this end, we note first that if $\alpha^* > 0$, ϕ_{α^*} and u_{α^*} (when it exists) are positive on $\bar{\Omega}$. Let $\alpha(u)$ be a density dependent choice of α which is nondecreasing, smooth and satisfies (1.2). Fix $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 < \alpha_2$ and suppose that

$$\lambda^1_{\alpha_2}(\Omega) < r/d \leqslant \lambda^1_{\alpha_1}(\Omega). \tag{1.15}$$

Then if $\alpha(u)$ satisfies

$$\alpha\Big(\min_{\bar{\Omega}} u_{\alpha_2}(r,d)\Big) > \alpha_2, \tag{1.16}$$

a solution $u(x,t)$ to (1.1) satisfies

$$u(x,t) > u_{\alpha_2}(r,d)(x)$$

for $x \in \bar{\Omega}$ and $t > 0$ provided that

$$u(x,0) > u_{\alpha_2}(r,d)(x)$$

for $x \in \bar{\Omega}$. If, in addition,

$$\alpha\Big(\varepsilon \max_{\bar{\Omega}} \phi_{\alpha_1}\Big) < \alpha_1 \tag{1.17}$$

for some $\varepsilon > 0$, a solution $u(x,t)$ to (1.1) tends to 0 over time provided that

$$u(x,0) < \varepsilon\phi_{\alpha_1}(x).$$

Consequently, if (1.16) and (1.17) hold, (1.1) admits an Allee effect for any (r,d) satisfying (1.15). It turns out that (1.16) and (1.17) can be satisfied for some α_1, α_2 provided $\alpha(1) = 1$ and $\alpha'(1)$ is small.

As noted, throughout the remainder of this article, we shall assume that either $\alpha(0) > 0$ or that $\alpha(0) = 0$ with $\alpha'(0) > 0$. In Section 2, we explore the possibility of obtaining an Allee effect in (1.1) on the basis of a local subcritical bifurcation of positive equilibria to (1.1) from the trivial equilibrium. We show that in the case when $\alpha(0) > 0$ such a phenomenon is indeed possible under suitable conditions on $\alpha'(0)$ and $\alpha''(0)$. In contrast, we show that when $\alpha(0) = 0$ with $\alpha'(0) > 0$, there can be no subcritical bifurcation of positive equilibria and hence no Allee effect based on subcritical bifurcation. In this case, solutions to the Dirichlet problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = d\nabla^2 u + ru(1-u) && \text{in } \Omega \times (0,\infty), \\ &u = 0 && \text{on } \partial\Omega \times (0,\infty) \end{aligned} \tag{1.18}$$

are always solutions to (1.1), and we show that the only positive equilibria to (1.1) that can emanate from the zero equilibrium are the positive equilibria to (1.18).

If we assume in addition that $\alpha(u)$ satisfies $\alpha(1) = 1$, we open the possibility of obtaining an Allee effect in (1.1) on the basis of having both the equilibria $u \equiv 0$ and $u \equiv 1$ locally asymptotically stable, as was the case in [5] when we assumed that $\alpha(u)$ satisfied (1.9). In Section 3, we make a local bifurcation analysis of the equilibria to (1.1) about $u \equiv 1$ in order to investigate this possibility. The only equilibria to (1.1) that can be asymptotic limits of positive solutions to (1.1) must take on values between 0 and 1 on Ω, and hence our primary interest is determining whether branches of such equilibria emanate from $u \equiv 1$. Recall that in this article we are assuming that $\alpha'(1) > 0$ when $\alpha(1) = 1$. This assumption creates a mathematical wrinkle in the analysis. Namely, if we consider the equilibrium problem corresponding to (1.1), write $u = 1 + v$, and linearize about $v \equiv 0$ we are led to the eigenvalue problem

$$\begin{aligned} &d\nabla^2\phi - r\phi = 0 && \text{in } \Omega, \\ &\nabla\phi \cdot \vec{\eta} - \alpha'(1)\phi = 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.19}$$

Since $\alpha'(1) > 0$, standard elliptic theory à la [9] does not apply to (1.19). We circumvent this difficulty by a change of variable of the form

$$w = hu, \tag{1.20}$$

where h in (1.20) is a suitably chosen positive function on $\bar{\Omega}$. In so doing, we obtain an equivalent reformulation of (1.1) to which we may apply the Crandall–Rabinowitz local bifurcation results [6]. In particular, we find that there is a unique and positive value of the ratio r/d for which (1.19) admits a positive solution and at which equilibrium solutions to (1.1) with values in $(0, 1)$ on Ω emanate from $u \equiv 1$. We then identify conditions on $\alpha(u)$ for which there is a range of values of the ratio r/d so that both $u \equiv 0$ and $u \equiv 1$ are locally asymptotically stable as equilibrium solutions of (1.1). For such values of r/d, (1.1) exhibits an Allee effect on the same basis as in [5] when $\alpha(u)$ satisfies (1.9).

Continuing to assume that $\alpha(1) = 1$, it is natural next to inquire as to the global structure of the branches of equilibria to (1.1) with values in $(0, 1)$ on Ω which emanate from $u \equiv 0$ and from $u \equiv 1$, and we turn to this topic in Section 4. Setting up a suitable functional analytic framework in which Rabinowitz's global bifurcation result [14] or one of its generalizations apply is somewhat complicated by the nonlinear nature of the boundary condition in (1.1). Moreover, we have found that the most advantageous means of setting up such a functional analytic framework differs depending upon which case was under consideration (i.e., when $\alpha(0) > 0$ versus when $\alpha(0) = 0$ with $\alpha'(0) > 0$).

When $\alpha(0) = 0$ and $\alpha'(0) > 0$, solutions to (1.18) are always solutions to (1.1). Consequently, the unbounded global branch of positive equilibria to (1.18) which emanates from $u \equiv 0$ (and which is a well-studied object [6]) must be a subset of the positive equilibria to (1.1). Indeed, as we have noted, we show in Section 2 that these are the only equilibria to (1.1) which emanate from the zero equilibrium. On the other hand, when $\alpha(0) = 0$ and $\alpha'(0) > 0$, equilibrium solutions to (1.1) which emanate from $u \equiv 1$ must necessarily satisfy (1.6). The results of Section 2 show that (except possibly when the spatial domain Ω is a ball) such equilibria are isolated in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ from the global branch of positive equilibria to (1.18) which emanates from $u \equiv 0$. As a result, the global branch of equilibria to (1.1) which emanates from $u \equiv 1$ with values in $(0, 1)$ on Ω must itself be unbounded in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ and must be distinct from the global branch of positive equilibrium solutions that emanates from $u \equiv 0$. In particular, we demonstrate in this case that for all large enough values of the ratio r/d, (1.1) has at least 3 equilibrium solutions with values in $(0, 1]$ on Ω (including $u \equiv 1$).

When $\alpha(0) > 0$, we show that there are unbounded continua of nontrivial equilibria to (1.1) with values in $[0, 1]$ on $\bar{\Omega}$ emanating from both $u \equiv 0$ and $u \equiv 1$. Clearly, an equilibrium solution to (1.1) with $u \equiv 1$ is "nontrivial" with respect to the equilibria with $u \equiv 0$. But notice that the reverse is also true; i.e., if the ray of equilibria $u \equiv 1$ represents the base or trivial branch of equilibria to (1.1) under consideration, then an equilibrium with $u \equiv 0$ is "nontrivial" with respect to this branch. As a result, there is no inconsistency with global bifurcation theory if the $u \equiv 0$ and $u \equiv 1$ equilibria to (1.1) are linked by equilibria with values in $(0, 1)$ on $\bar{\Omega}$. Indeed, it is entirely conceivable that if we regard the ratio r/d as the bifurcation parameter in our discussion, such a "linking set" of equilibria could have compact closure in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$. In Section 4, we show that there are $\alpha(u)$ with $\alpha(0) > 0$ so that for all large enough values of r/d, the only equilibria to (1.1) with values in $[0, 1]$ on $\bar{\Omega}$ are $u \equiv 0$ and $u \equiv 1$. For such $\alpha(u)$, the $u \equiv 0$ and $u \equiv 1$ branches of equilibria are thus linked in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ by a precompact (hence bounded in $\mathbb{R}$) continuum of equilibria to (1.1) with values in $(0, 1)$ on $\bar{\Omega}$. So for such $\alpha(u)$, the global structure of branches of equilibria to (1.1) is very different from the case when $\alpha(u)$ is such that $\alpha(0) = 0$ and $\alpha'(0) > 0$. We do not know at present whether such always is the case, since our proof places assumptions on $\alpha(u)$ beyond having $\alpha(0) > 0$.

Finally, in Section 5, we consider a side question that arises naturally from the discussion in Section 3. Recall from Section 3 that the only equilibria that can arise as asymptotic limits of nonnegative biologically relevant solutions to (1.1) must take values on $[0, 1]$ on $\bar{\Omega}$ and correspond to ratios of r/d with both $r \geqslant 0$ and $d > 0$. There is a unique and positive value of r/d at which such equilibria bifurcate from the equilibria with $u \equiv 1$ on $\bar{\Omega}$. This positive value of r/d is the unique value for which (1.19) admits an eigenfunction ϕ which is positive on $\bar{\Omega}$. Conceivably, there could be additional positive values of r/d for which (1.19) admits a nonzero solution. At such points there could be bifurcation of positive equilibria to (1.1) from $u \equiv 1$. These equilibria would take on values above and below 1 on $\bar{\Omega}$ and are not relevant from a biological point of view. However, they are part of the set of positive equilibria to (1.1), so it is of some interest to ask if there are additional positive values of r/d for which (1.19) admits a nonzero solution. We answer this question in Section 5 in the special case where Ω is an interval.

2. Possibility of Allee effects via subcritical bifurcation of equilibria from the zero state

In this section, we employ the classical local bifurcation results of Crandall and Rabinowitz (see, for example, [6, Chapter 3]) to explore the possibility of obtaining an Allee effect in (1.1) via a subcritical bifurcation of equilibrium solutions from the zero state. To this end, equilibrium solutions to (1.1) may be realized as the zeros of the mapping $F:\mathbb{R}\times C^{2,\gamma}(\bar{\Omega})\to C^{\gamma}(\bar{\Omega})\times C^{1,\gamma}(\partial\Omega)$ that is given by

$$F(\lambda,u)=\left(\nabla^2 u+\lambda u(1-u),\ \alpha(u)\nabla u\cdot\vec{\eta}+\left(1-\alpha(u)\right)u\right),\tag{2.1}$$

where $\lambda=r/d$. A straightforward calculation shows that the linearization of (2.1) with respect to u at $u=0$ is given by

$$F_u(\lambda,0)w=\left(\nabla^2 w+\lambda w,\ \alpha(0)\nabla w\cdot\vec{\eta}+\left(1-\alpha(0)\right)w\right).\tag{2.2}$$

By (1.2), $\alpha(0)\in[0,1]$. Suppose now that $\alpha(0)>0$. Then if $\lambda=\lambda^1_{\alpha(0)}(\Omega)$, where $\lambda^1_{\alpha(0)}(\Omega)$ is as in (1.13), it follows from the results of [9,11] that $F_u(\lambda,0)$ in (2.2) is a Fredholm operator of index 0 from $C^{2,\gamma}(\bar{\Omega})$ to $C^{\gamma}(\bar{\Omega})\times C^{1,\gamma}(\partial\Omega)$. Moreover, the kernel $N(F_u(\lambda,0))=\langle\phi\rangle$, where ϕ is an eigenfunction corresponding to $\lambda^1_{\alpha(0)}(\Omega)$ in (1.13) and $\phi(x)>0$ in Ω. (Note that $\phi(x)>0$ on $\bar{\Omega}$ as $\alpha(0)>0$.) Consequently, the co-dimension of the range $R(F_u(\lambda,0))$ in $C^{\gamma}(\bar{\Omega})\times C^{1,\gamma}(\partial\Omega)$ is 1 when $\lambda=\lambda^1_{\alpha(0)}(\Omega)$.

Hence we may employ the Crandall–Rabinowitz theorem to analyze the zeros of (2.1) in a neighborhood of $(\lambda^1_{\alpha(0)}(\Omega),0)$ provided that we show that $F_{\lambda u}(\lambda^1_{\alpha(0)}(\Omega),0)\phi\notin R(F_u(\lambda^1_{\alpha(0)}(\Omega),0))$. Now $F_{\lambda u}(\lambda^1_{\alpha(0)}(\Omega),0)\phi=(\phi,0)$, so we need to eliminate the possibility of the existence of a function $y\in C^{2,\gamma}(\bar{\Omega})$ so that

$$\begin{aligned}&\nabla^2 y+\lambda^1_{\alpha(0)}(\Omega)y=\phi &&\text{in }\Omega,\\ &\alpha(0)\nabla y\cdot\vec{\eta}+\left(1-\alpha(0)\right)y=0 &&\text{on }\partial\Omega.\end{aligned}\tag{2.3}$$

Should such a y exist, multiplying the top equation in (2.3) by ϕ, integrating and employing Green's second identity and (1.13) leads to the conclusion that

$$\int_{\Omega}\phi^2\,dx=0,$$

a contradiction. So the Crandall–Rabinowitz theorem applies. As a result, the zeros of (2.1) in a neighborhood of $(\lambda^1_{\alpha(0)}(\Omega),0)$ may be precisely described. Specifically, if any complement W of $\langle\phi\rangle$ in $C^{2,\gamma}(\bar{\Omega})$ is fixed, there are continuously differentiable functions $\lambda(s)$ and $\rho(s)$ defined on an open interval I_0 about 0 in $\mathbb{R}$ and mapping into $\mathbb{R}$ and W, respectively, with $\lambda(0)=\lambda^1_{\alpha(0)}(\Omega)$, $\rho(0)=0$ and

$$F\left(\lambda(s),s\phi+s\rho(s)\right)=0$$

for $s\in I_0$. Moreover, if $F(\lambda,y)=0$ and (λ,y) is sufficiently close to $(\lambda^1_{\alpha(0)}(\Omega),0)$, then either $y=0$ or else there is $s\neq 0\in I$ so that

$$(\lambda,y)=\left(\lambda(s),s\phi+s\rho(s)\right).\tag{2.4}$$

Suppose now that $\alpha(0)=0$. Then any solution of

$$\begin{aligned} &\nabla^2 y+\lambda y(1-y)=0 \quad && \text{in } \Omega, \\ &y=0 && \text{on } \partial\Omega \end{aligned} \tag{2.5}$$

is also a solution of

$$\begin{aligned} &\nabla^2 y+\lambda y(1-y)=0 && \text{in } \Omega, \\ &\alpha(y)\nabla y\cdot\vec{\eta}+\big(1-\alpha(y)\big)y=0 \quad && \text{on } \partial\Omega. \end{aligned} \tag{2.6}$$

When $\alpha(0)=0$ and $F:\mathbb{R}\times C^{2,\gamma}(\bar{\Omega})\to C^{\gamma}(\bar{\Omega})\times C^{1,\gamma}(\partial\Omega)$ is as in (2.1), the results of [9,11] guarantee that $F_u(\lambda,0)$ is a Fredholm operator of index 0. *However*, they show that it is Fredholm of index 0 as a map between $C^{2,\gamma}(\bar{\Omega})$ and $C^{\gamma}(\bar{\Omega})\times C^{2,\gamma}(\partial\Omega)$, *not* between $C^{2,\gamma}(\bar{\Omega})$ and $C^{\gamma}(\bar{\Omega})\times C^{1,\gamma}(\partial\Omega)$. Consequently, in this case, we cannot appeal to the Crandall–Rabinowitz theorem to assert that the only positive solutions to (2.6) in a neighborhood of $(\lambda^1_{\alpha(0)}(\Omega),0)$ are solutions of (2.5). Fortunately, we can establish this fact directly. To this end, suppose there is a sequence $\{(\lambda_n,u_n)\}\subseteq\mathbb{R}\times C^{2,\gamma}(\bar{\Omega})$ with $\lambda_n>0$, $u_n>0$ in Ω so that

$$\begin{aligned} &-\nabla^2 u_n=\lambda_n u_n(1-u_n) && \text{in } \Omega, \\ &\alpha(u_n)\nabla u_n\cdot\vec{\eta}+\big(1-\alpha(u_n)\big)u_n=0 \quad && \text{on } \partial\Omega, \end{aligned}$$

where

$$\alpha(u_n)\not\equiv 0 \quad \text{on } \partial\Omega$$

and $(\lambda_n,u_n)\to(\bar{\lambda},0)$ in $\mathbb{R}\times C^{2,\gamma}(\bar{\Omega})$ for some $\bar{\lambda}\geqslant 0$. As in (1.7) we assume $\alpha'(0)>0$. Then for each n, there is an $x_n\in\partial\Omega$ so that $\alpha(u_n(x_n))\neq 0$, which implies $u_n(x_n)\neq 0$. So we have

$$\nabla u_n(x_n)\cdot\vec{\eta}=\frac{\alpha(u_n(x_n))-1}{\beta(u_n(x_n))} \tag{2.7}$$

for all n, where β is as in (1.5). Since $u_n\to 0$ in $C^{2,\gamma}(\bar{\Omega})$, the left-hand side of (2.7) must converge to 0. However, the right-hand side converges to $-\frac{1}{\beta(0)}<0$, a contradiction. Consequently, there can be no such sequence. Hence the only solutions to (2.6) which can bifurcate from 0 in this case are solutions to (2.5). It is well known [6] that if $\lambda>\lambda^1_0(\Omega)$, there is a unique positive solution $\bar{u}(\lambda)$ of (2.5) in $C^{2,\gamma}(\bar{\Omega})$ and that the map $\lambda\to\bar{u}(\lambda)$ is smooth from $(\lambda^1_0(\Omega),\infty)$ to $C^{2,\gamma}(\bar{\Omega})$ with $\lim_{\lambda\to\lambda^1_0(\Omega)^+}\bar{u}(\lambda)=0$. As a result, there is not a subcritical bifurcation of equilibria to (1.1) in this case. (Of course, more is known about $\bar{u}(\lambda)$. Namely, if one considers the model

$$\begin{aligned} &\frac{\partial u}{\partial t}=\nabla^2 u+\lambda u(1-u) \quad && \text{in } \Omega\times(0,\infty), \\ &u=0 && \text{on } \partial\Omega\times(0,\infty), \end{aligned} \tag{2.8}$$

any solution $u(x,t)$ with $u(x,0)\overset{>}{\not\equiv}0$ converges over time to $\bar{u}(\lambda)$ in $C^{1,\gamma}(\bar{\Omega})$ if $\lambda>\lambda^1_0(\Omega)$ and to 0 if $\lambda\leqslant\lambda^1_0(\Omega)$.)

The preceding observations show that if $\alpha(0)=0$, and $\alpha'(0)>0$, there cannot be an Allee effect in (1.1) resulting from a subcritical bifurcation of equilibria from the zero state. In fact, they show more. Namely, for any bounded interval $[a,b]\subset(0,\infty)$, there must be an $\varepsilon=\varepsilon([a,b])>0$ so that there is no $(\lambda,u)\in[a,b]\times C^{2,\gamma}(\bar\Omega)$ solving (2.6) with $\lambda\in[a,b]$, $u>0$ in Ω, $\alpha(u)\not\equiv 0$ on $\partial\Omega$ and $\|u\|_{C^{2,\gamma}(\bar\Omega)}<\varepsilon$. So the only small norm positive equilibria to (1.1) when $\alpha(0)=0$ are those along $\{(\lambda,\bar u(\lambda))\colon \lambda>\lambda_0^1(\Omega)\}$ with λ near $\lambda_0^1(\Omega)$.

It is natural to ask whether the set $\{(\lambda,\bar u(\lambda))\colon \lambda>\lambda_0^1(\Omega)\}$ is globally isolated as a subset of the solution set to (2.6). To address this issue, first recall from the introduction that (1.1) is ill-posed when $\alpha(0)=0$. However, recall also that when $\alpha'(0)>0$, any solution $u(x,t)$ of (1.1) with $u(x,0)>0$ on $\bar\Omega$ necessarily satisfies

$$\beta(u)\nabla u\cdot\vec\eta+\bigl(1-\alpha(u)\bigr)=0 \tag{2.9}$$

on $\partial\Omega\times(0,\infty)$, where

$$\beta(u)=\begin{cases}\frac{\alpha(u)}{u}, & u\neq 0,\\ \alpha'(0), & u=0.\end{cases}$$

Consequently, if $\alpha'(0)>0$, any equilibrium solution to (1.1) which could be in the omega limit set of such a solution to (1.1) must satisfy (2.9). So in considering whether the set $\mathcal{C}=\{(\lambda,\bar u(\lambda))\colon \lambda>\lambda_0^1(\Omega)\}$ is isolated as a subset of the solutions to (2.6), we will continue to assume $\alpha'(0)>0$ and only consider solutions to (2.6) which are such that $u\equiv 0$ on $\partial\Omega$ or which satisfy (2.9) on $\partial\Omega$. We already know that $\mathcal{C}$ is isolated relative to solutions to (2.6) which vanish on $\partial\Omega$. So now suppose that $\lambda>\lambda_0^1(\Omega)$, and that there is a sequence $\{(\lambda_n,w_n)\}$ in $\mathbb{R}\times C^{1,\gamma}(\bar\Omega)$ converging to $(\lambda,\bar u(\lambda))$ in $\mathbb{R}\times C^{1,\gamma}(\bar\Omega)$ with

$$\begin{aligned}&\nabla^2 w_n+\lambda_n w_n(1-w_n)=0 && \text{in } \Omega,\\ &\beta(w_n)\nabla w_n\cdot\vec\eta+\bigl(1-\alpha(w_n)\bigr)=0 && \text{on } \partial\Omega.\end{aligned} \tag{2.10}$$

Since $w_n\to\bar u(\lambda)$ in $C^{1,\gamma}(\bar\Omega)$ as $n\to\infty$, $w_n\to 0$ in $C^{1,\gamma}(\partial\Omega)$ and $\nabla w_n\cdot\vec\eta\to\nabla\bar u(\lambda)\cdot\vec\eta$ in $C^{\gamma}(\partial\Omega)$ as $n\to\infty$. As a consequence, it follows from the second equation in (2.10) that

$$\alpha'(0)\nabla\bar u(\lambda)\cdot\vec\eta+1=0 \tag{2.11}$$

on $\partial\Omega$. So from (2.11), we have that for such a λ, $\bar u(\lambda)$ and $\nabla\bar u(\lambda)\cdot\vec\eta$ are constant on $\partial\Omega$. A result of Serrin [15] tells us that Ω is a ball. So provided that Ω is not a ball, $\{(\lambda,\bar u(\lambda))\colon \lambda>\lambda_0^1(\Omega)\}$ is isolated among the solutions to (2.6) globally in the sense described above.

If Ω is a ball, it is sometimes possible to have a positive solution of (2.6) which vanishes on $\partial\Omega$ and also satisfies (2.9), meaning that our restriction to cases in which Ω is not a ball is not artificial. To see that such is the case, take $\Omega=(0,1)$ and let u be a positive solution of (2.5) for some positive value of λ. There is then a $\gamma>0$ so that $\nabla u\cdot\vec\eta=-\gamma$ at $x=0$ and $x=1$. Provided β in (2.9) is such that $\beta(0)=1/\gamma$, u also satisfies (2.9).

We now know that in order for an Allee effect to arise in (1.1) via a subcritical bifurcation of equilibria from the zero state when $\lambda=\lambda_{\alpha(0)}^1(\Omega)$, we must require $\alpha(0)>0$. So we assume

$$\alpha(0)>0. \tag{2.12}$$

A subcritical bifurcation of equilibria to (1.1) boils down to having $\lambda'(0) < 0$, where $\lambda(s)$ is as given in (2.4). The Crandall–Rabinowitz theorem justifies differentiating the expressions in (2.1) with respect to s for s near 0. Indeed, since F in (2.1) is differentiable to any order in λ and in u, we may take as many derivatives with respect to s as is necessary. Our calculation of $\lambda'(0)$ will require differentiating (2.1) with respect to s twice. To this end, let $' = d/ds$ and $'' = d^2/ds^2$. Then differentiating $F(\lambda, u) = 0$, where F is as in (2.1), once yields

$$\nabla^2 u' + \lambda'(u - u^2) + \lambda(1 - 2u)u' = 0 \quad \text{in } \Omega, \tag{2.13}$$

$$\frac{d}{du}\big(\alpha(u)\big)u'\nabla u \cdot \vec{\eta} + \alpha(u)\nabla u' \cdot \vec{\eta} - \frac{d}{du}\big(\alpha(u)\big)u'u + \big(1 - \alpha(u)\big)u' = 0 \quad \text{on } \partial\Omega, \tag{2.14}$$

a second differentiation produces

$$\nabla^2 u'' + \lambda''(u - u^2) + 2\lambda'(1 - 2u)u' - 2\lambda(u')^2 + \lambda(1 - 2u)u'' = 0 \quad \text{in } \Omega, \tag{2.15}$$

$$\begin{aligned} &\frac{d^2(\alpha(u))}{du^2}(u')^2\nabla u \cdot \vec{\eta} + \frac{d(\alpha(u))}{du}u''\nabla u \cdot \vec{\eta} + 2\frac{d}{du}\big(\alpha(u)\big)u'\nabla u' \cdot \vec{\eta} + \alpha(u)\nabla u'' \cdot \vec{\eta} \\ &\quad - \frac{d^2}{du^2}\big(\alpha(u)\big)(u')^2u - 2\frac{d}{du}\big(\alpha(u)\big)(u')^2 - \frac{d}{du}\big(\alpha(u)\big)u''u + \big(1 - \alpha(u)\big)u'' = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.16}$$

When $s = 0$, $u = 0$, $u' = \phi$ and $\lambda = \lambda^1_{\alpha(0)}(\Omega)$. Consequently (2.13)–(2.14) and (2.15)–(2.16) reduce to

$$\nabla^2\phi + \lambda^1_{\alpha(0)}(\Omega)\phi = 0 \quad \text{in } \Omega, \tag{2.17}$$

$$\alpha(0)\nabla\phi \cdot \vec{\eta} + \big(1 - \alpha(0)\big)\phi = 0 \quad \text{on } \partial\Omega \tag{2.18}$$

and

$$\nabla^2 u'' + 2\lambda'\phi - 2\lambda^1_{\alpha(0)}(\Omega)\phi^2 + \lambda^1_{\alpha(0)}(\Omega)u'' = 0 \quad \text{in } \Omega, \tag{2.19}$$

$$2\frac{d\alpha}{du}(0)\phi\nabla\phi \cdot \vec{\eta} + \alpha(0)\nabla u'' \cdot \vec{\eta} - 2\frac{d\alpha}{du}(0)\phi^2 + \big(1 - \alpha(0)\big)u'' = 0 \quad \text{on } \partial\Omega, \tag{2.20}$$

respectively.

To determine $\lambda'(0)$, we multiply (2.19) by ϕ and integrate, obtaining

$$\int_\Omega \phi\nabla^2 u''\,dx + 2\lambda'(0)\int_\Omega \phi^2\,dx - 2\lambda^1_{\alpha(0)}(\Omega)\int_\Omega \phi^3\,dx + \lambda^1_{\alpha(0)}(\Omega)\int_\Omega u''\phi\,dx = 0. \tag{2.21}$$

Now

$$
\begin{aligned}
\int_{\Omega} \phi \nabla^2 u'' \, dx &= \int_{\Omega} u'' \nabla^2 \phi \, dx + \int_{\partial\Omega} (\phi \nabla u'' \cdot \vec{\eta} - u'' \nabla \phi \cdot \vec{\eta}) \, dS \\
&= -\lambda^1_{\alpha(0)}(\Omega) \int_{\Omega} u'' \phi \, dx \\
&\quad + \int_{\partial\Omega} \left\{ \frac{\phi}{\alpha(0)} \left[2\frac{d\alpha}{du}(0)\phi^2 - 2\frac{d\alpha(0)}{du}\phi \nabla\phi \cdot \vec{\eta} - \big(1-\alpha(0)\big)u'' \right] \right. \\
&\quad \left. - \frac{u''}{\alpha(0)}\big[-\big(1-\alpha(0)\big)\phi\big] \right\} dS \\
&= -\lambda^1_{\alpha(0)}(\Omega) \int_{\Omega} u'' \phi \, dx + \frac{2\frac{d\alpha}{du}(0)}{\alpha(0)} \int_{\partial\Omega} \phi^3 \left(1 + \frac{1-\alpha(0)}{\alpha(0)}\right) dS \\
&= -\lambda^1_{\alpha(0)}(\Omega) \int_{\Omega} u'' \phi \, dx + \frac{2\frac{d\alpha}{du}(0)}{(\alpha(0))^2} \int_{\partial\Omega} \phi^3 \, dS
\end{aligned}
$$

by Green's second identity and (2.17)–(2.20). Substituting into (2.21) yields

$$
2\lambda'(0) \int_{\Omega} \phi^2 \, dx - 2\lambda^1_{\alpha(0)}(\Omega) \int_{\Omega} \phi^3 \, dx + \frac{2\frac{d\alpha}{du}(0)}{(\alpha(0))^2} \int_{\partial\Omega} \phi^3 \, dS = 0
$$

so that

$$
\lambda'(0) = \frac{\lambda^1_{\alpha(0)}(\Omega) \int_{\Omega} \phi^3 \, dx - \frac{\frac{d\alpha}{du}(0)}{(\alpha(0))^2} \int_{\partial\Omega} \phi^3 \, dS}{\int_{\Omega} \phi^2 \, dx}. \tag{2.22}
$$

It is important to note that $\lambda^1_{\alpha(0)}(\Omega)$ and ϕ depend on the value of $\alpha(0)$ but are independent of $\frac{d\alpha}{du}(0)$. Consequently, it follows from (2.22) that $\lambda'(0) < 0$ provided

$$
\frac{d\alpha}{du}(0) > \big(\alpha(0)\big)^2 \lambda^1_{\alpha(0)}(\Omega) \frac{\int_{\Omega} \phi^3 \, dx}{\int_{\partial\Omega} \phi^3 \, ds}. \tag{2.23}
$$

Under assumption (2.23), there is a subcritical bifurcation of equilibria to (1.1) at $(\lambda^1_{\alpha(0)}(\Omega), 0)$. To have the possibility of obtaining an Allee effect in (1.1) from this phenomenon, we need to know two things. Namely, we need to know first of all that the zero equilibrium to (1.1) is stable for $0 < \lambda < \lambda^1_{\alpha(0)}(\Omega)$ and unstable for $\lambda > \lambda^1_{\alpha(0)}(\Omega)$. Secondly, we need to know that the equilibrium $(\lambda(s), s(\phi + \rho(s)))$ is unstable for $s > 0$ and small. We may demonstrate both via the method of upper and lower solutions in a manner analogous to [4], provided we can appeal to the extension [8] of the Crandall–Rabinowitz theory which relates the linearized stability/instability of the zero equilibria to (1.1) to that along the bifurcating branch $(\lambda(s), s(\phi + \rho(s)))$ for s small (i.e., near the bifurcation point). In essence, the Crandall–Rabinowitz result establishes that having a subcritical bifurcation (in the λ parameter) from the branch of trivial solutions is equivalent to the bifurcating branch of nonzero solutions having the opposite linearized stability (when viewed as equilibria to the corresponding dynamical problem) as the trivial solutions along the direction of bifurcation in λ. Such an appeal is valid in our case, since the map $u \to (u, 0)$ is a continuous embedding of $C^{2,\gamma}(\bar{\Omega})$ into $C^{\gamma}(\bar{\Omega}) \times C^{1,\gamma}(\partial\Omega)$ that makes it possible to consider eigenvalue problems of the form

$$F_u(\lambda, u)z = \sigma z, \tag{2.24}$$

where F is as in (2.1).

Under assumption (2.23), it follows from formula (1.17) of [8] that the principal eigenvalue σ in (2.24) is positive for the nonzero equilibrium solutions (λ, u) of (1.1) which are sufficiently close to $(\lambda^1_{\alpha(0)}(\Omega), 0)$ in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$. Hence such solutions are linearly unstable. To obtain a more refined picture of the instability, let z be an eigenfunction corresponding to σ, so that z can be assumed to be positive in $\bar{\Omega}$ and to satisfy

$$\begin{aligned} &\nabla^2 z + \lambda(1-2u)z = \sigma z && \text{in } \Omega, \\ &\alpha(u)\nabla z \cdot \vec{\eta} + \alpha'(u)\nabla u \cdot \vec{\eta} z - \alpha'(u)uz + \big(1-\alpha(u)\big)z = 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.25}$$

For such a z, it is easy to show that

$$\nabla^2(u+\varepsilon z) + \lambda(u+\varepsilon z)\big(1-(u+\varepsilon z)\big) > 0 \tag{2.26}$$

on Ω. On $\partial\Omega$, one has that

$$\begin{aligned} &\nabla(u+\varepsilon z)\cdot\vec{\eta} + \left(\frac{1}{\alpha(u+\varepsilon z)} - 1\right)(u+\varepsilon z) \\ &= -\left(\frac{1}{\alpha(u)} - 1\right)u + \varepsilon\left[-\left(\frac{1}{\alpha(u)} - 1\right)z + \frac{\alpha'(u)}{\alpha(u)}uz - \frac{\alpha'(u)}{\alpha(u)}\nabla u\cdot\vec{\eta}z\right] \\ &\quad + \left(\frac{1}{\alpha(u+\varepsilon z)} - 1\right)(u+\varepsilon z) \\ &= \left(\frac{1}{\alpha(u+\varepsilon z)} - \frac{1}{\alpha(u)}\right)(u+\varepsilon z) + \frac{\alpha'(u)}{\alpha(u)}u\varepsilon z - \frac{\alpha'(u)}{\alpha(u)}\nabla u\cdot\vec{\eta}\varepsilon z \\ &= \left(\frac{1}{\alpha(u+\varepsilon z)} - \frac{1}{\alpha(u)}\right)(u+\varepsilon z) + \frac{\alpha'(u)}{\alpha(u)}u\varepsilon z + \frac{\alpha'(u)}{\alpha(u)}\left(\frac{1}{\alpha(u)} - 1\right)u\varepsilon z \\ &= \left(\frac{1}{\alpha(u+\varepsilon z)} - \frac{1}{\alpha(u)}\right)(u+\varepsilon z) + \frac{\alpha'(u)}{[\alpha(u)]^2}u\varepsilon z. \end{aligned} \tag{2.27}$$

Now by the mean value theorem there are ρ and $\tilde{\rho}$ with $u < \tilde{\rho} < \rho < u + \varepsilon z$ so that the last formula in (2.27) becomes

$$\begin{aligned} &= \left(\frac{1}{\alpha(u)} - \frac{\alpha'(\rho)}{[\alpha(\rho)]^2}\varepsilon z - \frac{1}{\alpha(u)}\right)(u+\varepsilon z) + \frac{\alpha'(u)}{[\alpha(u)]^2}u\varepsilon z \\ &= -\frac{\alpha'(\rho)}{[\alpha(\rho)]^2}u\varepsilon z - \frac{\alpha'(\rho)}{[\alpha(\rho)]^2}\varepsilon^2 z^2 + \frac{\alpha'(u)}{[\alpha(u)]^2}u\varepsilon z \end{aligned}$$

$$= \left[\frac{\alpha'(u)}{[\alpha(u)]^2} - \frac{\alpha'(\rho)}{[\alpha(\rho)]^2}\right]u\varepsilon z - \frac{\alpha'(\rho)}{[\alpha(\rho)]^2}\varepsilon^2 z^2$$

$$= \left(\frac{2\alpha'(\tilde{\rho})^2 - \alpha(\tilde{\rho})\alpha''(\tilde{\rho})}{[\alpha(\tilde{\rho})]^3}\right)(u-\rho)u\varepsilon z - \frac{\alpha'(\rho)}{[\alpha(\rho)]^2}\varepsilon^2 z^2. \tag{2.28}$$

If now

$$2[\alpha'(0)]^2 - \alpha(0)\alpha''(0) > 0, \tag{2.29}$$

we have that (2.27) (or (2.28)) is negative on $\partial\Omega$ provided (λ, u) is close enough to $(\lambda^1_{\alpha(0)}(\Omega), 0)$ in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ and $\varepsilon > 0$ is sufficiently small. Consequently, if (2.23) and (2.29) hold, $u + \varepsilon\phi$ is a strict lower solution for $F(\lambda, u) = 0$, where F is as in (2.1) provided that (λ, u) is an equilibrium for (1.1) sufficiently close to $(\lambda^1_{\alpha(0)}(\Omega), 0)$ in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ and $\varepsilon > 0$ is sufficiently small.

One may show that the semi-flow associated to (1.1) is order-preserving for u values in $(0, 1)$. As a result, one may proceed as in [4] to show that if (λ, u) and ε are as in the preceding paragraph and w is the solution of (1.1) with $w(x, 0) = u(x) + \varepsilon z(x)$, then $w(x, t)$ is an increasing function in t and converges to an equilibrium solution of (1.1) as $t \to \infty$. In particular, it follows that any solution of (1.1) with initial data exceeding $u(x)$ must exceed $u(x)$ for all positive time. Consequently, if both (2.23) and (2.29) hold, we may employ the method of upper and lower solutions to obtain a threshold effect for solutions to (1.1) with initial data near $(\lambda^1_{\alpha(0)}(\Omega), 0)$. As previously noted, the linearized instability of points along the branch of nonzero equilibria to (1.1) which emanates from $(\lambda^1_{\alpha(0)}(\Omega), 0)$ may be deduced solely on the basis of (2.23). We employ (2.29) in order to get a more refined description of this instability via the method of upper and lower solutions. Whether (2.29) (which depends on $\alpha''(0)$ as well as on $\alpha'(0)$ and $\alpha(0)$) is a necessary condition to obtain this threshold effect is an interesting open question.

Summing up, the analysis in this section shows that obtaining an Allee effect in the solutions to (1.1) via a subcritical bifurcation of equilibria to (1.1) from the set of trivial solutions is only possible when the function $\alpha(u)$ satisfies $\alpha(0) > 0$. In this case, subcritical bifurcation of equilibria for (1.1) at $(\lambda^1_{\alpha(0)}(\Omega), 0)$ occurs when (2.23) holds. When the additional condition (2.29) is assumed, we obtain a threshold effect among solutions to (1.1) via the method of upper and lower solutions. We shall continue discussion of Allee effects and multiple equilibria over the next two sections of the paper.

3. Bifurcation from $u \equiv 1$: Multiple equilibria and Allee effects

Let us now assume that $\alpha(1) = 1$, so that $u \equiv 1$ is an equilibrium solution to (1.1) for all choices of $d > 0$ and $r > 0$. Recall from our discussion in Section 1 that we shall also assume that $\alpha'(1) > 0$ when $\alpha(1) = 1$. As in the previous section, let $\lambda = r/d$ and consider $F(\lambda, u) = 0$, where F is as in (2.1); i.e.,

$$\begin{aligned} -\nabla^2 u &= \lambda u(1-u) && \text{in } \Omega, \\ \alpha(u)\nabla u \cdot \vec{\eta} + \big(1 - \alpha(u)\big)u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

In order to look for bifurcation from $u \equiv 1$, one should write $u = 1 + v$ and substitute into (3.1), leading to

$$\begin{aligned} -\nabla^2 v &= -\lambda v(1+v) && \text{in } \Omega, \\ \alpha(1+v)\nabla v \cdot \vec{\eta} + \big(1 - \alpha(1+v)\big)(1+v) &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Bifurcation from $u \equiv 1$ in (3.1) corresponds to bifurcation from $v \equiv 0$ in (3.2). To apply the Crandall–Rabinowitz theorem, one would then need to calculate and examine the linearization of (3.2) about $v \equiv 0$, which turns out to be

$$\begin{aligned} -\nabla^2 \phi &= -\lambda\phi && \text{in } \Omega, \\ \nabla\phi \cdot \vec{\eta} - \alpha'(1)\phi &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

Since $\alpha'(1) > 0$, the Robin boundary condition in (3.3) "goes the wrong way" and hence is such that standard elliptic theory (e.g., [9]) no longer applies. Alternatively, problem (3.3) could be regarded as a Stekloff problem [3], in which $\alpha'(1)$ is the principal Stekloff eigenvalue for the operator $\nabla^2 - \lambda I$. However, in our treatment, it is more suitable to think of $\alpha'(1)$ as a given quantity and $\lambda = r/d$ as the parameter of primary interest, so we shall not pursue this approach further.

Consequently, we are not able to verify the hypotheses of the Crandall–Rabinowitz theorem if we simply proceed in the most direct route toward bifurcation from $u \equiv 1$ in (3.1). To circumvent the problem, we make an initial change of variables

$$w = hu, \tag{3.4}$$

where $h = h(x)$ is a positive function on $\bar{\Omega}$ that we shall specify a bit later. (Here our approach was inspired by Protter and Weinberger's approach to establishing the generalized maximum principle [13], although our focus is substantively different.) A straightforward calculation via (3.4) shows that the first equation in (3.1) can be expressed in terms of w as

$$\frac{1}{h}\nabla^2 w - \frac{2\nabla h \cdot \nabla w}{h^2} - \left(\frac{\nabla^2 h}{h^2} - \frac{2|\nabla h|^2}{h^3}\right) w + \lambda\left(\frac{w}{h} - \frac{w^2}{h^2}\right) = 0$$

in Ω, which upon division by h yields

$$\nabla \cdot \frac{1}{h^2}\nabla w - \left(\frac{\nabla^2 h}{h^3} - \frac{2|\nabla h|^2}{h^4}\right) w + \lambda\left(\frac{w}{h^2} - \frac{w^2}{h^3}\right) = 0 \tag{3.5}$$

in Ω. Likewise, the boundary condition in (3.1) in terms of w is

$$\alpha\left(\frac{w}{h}\right)\nabla w \cdot \vec{\eta} - \left[\alpha\left(\frac{w}{h}\right)\frac{\nabla h \cdot \vec{\eta}}{h} - \left(1 - \alpha\left(\frac{w}{h}\right)\right)\right] w = 0 \tag{3.6}$$

on $\partial\Omega$. In order for the boundary condition to "go the right way" in our context, we need

$$\alpha\left(\frac{w}{h}\right)\frac{\nabla h \cdot \vec{\eta}}{h} - \left(1 - \alpha\left(\frac{w}{h}\right)\right) < 0 \tag{3.7}$$

for all $w \geqslant 0$. Since (3.7) can be re-written as

$$\alpha\left(\frac{w}{h}\right)\left[\frac{\nabla h\cdot\vec{\eta}}{h}+1\right]<1,$$

(3.7) holds so long as

$$\frac{\nabla h\cdot\vec{\eta}}{h}=-K \tag{3.8}$$

for any $K>1$. (If $\partial\Omega$ is smooth, it is always possible to choose such an h for a given K.)

Assuming (3.8), we express (3.1) in terms of w as (3.5)–(3.6). Then $w\equiv h$ in (3.5)–(3.6) corresponds to $u\equiv 1$ in (3.1). So now write $w=\rho+h$. Let $V\subseteq C^{2,\gamma}(\bar{\Omega})$ be a neighborhood of 0 which is such that if $p\in V$, $p+h>0$ in $\bar{\Omega}$. Define $G:\mathbb{R}\times V\to C^{\gamma}(\bar{\Omega})\times C^{1,\gamma}(\partial\Omega)$ by

$$\begin{aligned}
G(\lambda,\rho)={}&\Bigg(\nabla\cdot\frac{1}{h^2}\nabla(\rho+h)-\left(\frac{\nabla^2 h}{h^3}-\frac{2|\nabla h|^2}{h^4}\right)(\rho+h)\\
&+\lambda\left(\frac{\rho+h}{h^2}-\frac{(\rho+h)^2}{h^3}\right),\ \alpha\left(\frac{\rho+h}{h}\right)\nabla(\rho+h)\cdot\vec{\eta}\\
&-\left[\alpha\left(\frac{\rho+h}{h}\right)\frac{\nabla h\cdot\vec{\eta}}{h}-\left(1-\alpha\left(\frac{\rho+h}{h}\right)\right)\right](\rho+h)\Bigg)\\
={}&\Bigg(\nabla\cdot\frac{1}{h^2}\nabla\rho-\left(\frac{\nabla^2 h}{h^3}-\frac{2|\nabla h|^2}{h^4}\right)\rho\\
&+\lambda\left(\frac{\rho}{h^2}-\frac{(\rho^2+2\rho h)}{h^3}\right),\ \alpha\left(\frac{\rho+h}{h}\right)\nabla(\rho+h)\cdot\vec{\eta}\\
&-\left[\alpha\left(\frac{\rho+h}{h}\right)\frac{\nabla h\cdot\vec{\eta}}{h}-\left(1-\alpha\left(\frac{\rho+h}{h}\right)\right)\right](p+h)\Bigg).
\end{aligned} \tag{3.9}$$

It follows from (3.5)–(3.6) that $G(\lambda,\rho)=(0,0)$ in (3.9) if and only if $F(\lambda,\frac{\rho+h}{h})=(0,0)$ in (2.1). In particular, we have that

$$G(\lambda,0)=(0,0) \tag{3.10}$$

for all $\lambda\in\mathbb{R}$.

Our aim now is to establish bifurcation from $u\equiv 1$ in (3.1) by applying the Crandall–Rabinowitz theorem to $G(\lambda,\rho)$. (At an appropriate point in our verification of the hypotheses of the Crandall–Rabinowitz theorem, we will identify a suitable choice of h satisfying (3.8).) Now

$$\begin{aligned}
G_\rho(\lambda,\rho)z={}&\Bigg(\nabla\cdot\frac{1}{h^2}\nabla z-\left(\frac{\nabla^2 h}{h^3}-\frac{2|\nabla h|^2}{h^4}\right)z\\
&+\lambda\left(\frac{1}{h^2}z-\frac{1}{h^3}(2\rho+2h)z\right),\ \alpha'\left(\frac{\rho+h}{h}\right)\frac{z}{h}\nabla(\rho+h)\cdot\vec{\eta}+\alpha\left(\frac{\rho+h}{h}\right)\nabla z\cdot\vec{\eta}
\end{aligned}$$

$$-\left[\alpha'\left(\frac{\rho+h}{h}\right)\frac{z}{h}\frac{\nabla h\cdot\vec{\eta}}{h}+\alpha'\left(\frac{\rho+h}{h}\right)\frac{z}{h}\right](\rho+h)$$
$$-\left[\alpha\left(\frac{\rho+h}{h}\right)\frac{\nabla h\cdot\vec{\eta}}{h}-\left(1-\alpha\left(\frac{\rho+h}{h}\right)\right)\right]z\bigg),$$

which when $\rho=0$ yields

$$\left(\nabla\cdot\frac{1}{h^2}\nabla z-\left(\frac{\nabla^2 h}{h^3}-\frac{2|\nabla h|^2}{h^4}\right)z-\lambda\frac{z}{h^2},\ \nabla z\cdot\eta-\left[\alpha'(1)+\frac{\nabla h\cdot\vec{\eta}}{h}\right]z\right). \tag{3.11}$$

We noted earlier that if $\frac{\nabla h\cdot\vec{\eta}}{h}=-K$, where $K>1$, the boundary condition "goes the right way" in the nonlinear problem for any $w\geqslant 0$. Consequently, such is the case in $G(\lambda,\rho)$ for all $\rho\in V$ provided $h>0$ on $\bar{\Omega}$. Such an h can be obtained, for example, as the solution of the boundary value problem

$$\begin{aligned} -\nabla^2 h &= 1 && \text{in } \Omega,\\ \nabla h\cdot\vec{\eta}+Kh &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.12}$$

where $K>1$. In light of (3.11), we assume in addition that

$$K>\alpha'(1). \tag{3.13}$$

Assuming (3.13), then $G_\rho(\lambda,0)$ in (3.11) is a Fredholm operator of index 0 from $C^{2,\gamma}(\bar{\Omega})$ to $C^\gamma(\bar{\Omega})\times C^{1,\gamma}(\partial\Omega)$ for all $\lambda\in\mathbb{R}$. Moreover, standard elliptic theory [7,9] now implies that the set of λ for which the null space

$$N\big(G_\rho(\lambda,0)\big)\neq\{0\} \tag{3.14}$$

is a decreasing sequence λ_n, $n\geqslant 0$, with $\lim_{n\to\infty}\lambda_n=-\infty$. Moreover, $N(G_\rho(\lambda_0,0))=\langle z\rangle$, where z may be chosen positive on $\bar{\Omega}$ and λ_0 is the only member of the collection $\{\lambda_n\colon n\geqslant 0\}$ admitting such a z.

Since $G_\rho(\lambda_0,0)$ is Fredholm of index 0, the first condition of the Crandall–Rabinowitz theorem is met when $\lambda=\lambda_0$. Of course, in our application λ corresponds to $r/d\geqslant 0$. So obtaining bifurcation from the line $\{(\lambda,0)\}$ of trivial solutions to $G(\lambda,\rho)=0$ is only of interest to us if $\lambda_0\geqslant 0$. Of course, when $\lambda_0>0$, there may be a finite number of additional values $\lambda_1,\dots,\lambda_m$ so that (3.14) holds and $\lambda_m\geqslant 0$. (We will examine this interesting side question in Section 5 in the case wherein Ω is an interval.) Potentially, there could be bifurcation from $u\equiv 1$ in (3.1) at these additional values. However, the maximum principle tells us that any equilibrium solution to (1.1) which is an asymptotic temporal limit of a positive solution to (1.1) must be less than or equal to 1 throughout Ω. Translated to $G(\lambda,\rho)=0$, such an equilibrium solutions corresponds to a $\rho\in(-h,0)$. Since λ_0 is the only λ_n so that $N(G_\rho(\lambda_n,0))$ contains a function which is of one sign on $\bar{\Omega}$, λ_0 is the only value of λ so that $(\lambda,0)$ could be the limit of a sequence of solutions to $G(\lambda,\rho)=0$ with $-h<\rho<0$. So we will only look for bifurcation from $u\equiv 1$ in (3.1) at $\lambda=\lambda_0$.

In order to employ the Crandall–Rabinowitz theorem to assert bifurcation of equilibria to (1.1) from $(\lambda, 1)$, it remains to establish that $\lambda_0 > 0$ and that if $N(G_\rho(\lambda_0, 0)) = \langle z \rangle$ with $z > 0$ in $\bar{\Omega}$, then

$$G_{\lambda\rho}(\lambda_0, 0)z \notin R\big(G_\rho(\lambda_0, 0)\big). \tag{3.15}$$

Assume for the moment that $\lambda_0 > 0$. It is not difficult to establish that

$$G_{\lambda\rho}(\lambda, \rho)v = \left(-\frac{1}{h^2}v - \frac{2\rho}{h^3}v, 0\right)$$

so that

$$G_{\lambda\rho}(\lambda_0, 0)z = \left(-\frac{1}{h^2}z, 0\right). \tag{3.16}$$

If (3.15) fails, it follows from (3.16) that there is a $y \in C^{2,\gamma}(\bar{\Omega})$ so that

$$\begin{aligned} &\nabla \cdot \frac{1}{h^2}\nabla y - \left(\frac{\nabla^2 h}{h^3} - \frac{2|\nabla h|^2}{h^4}\right)y - \lambda_0\left(\frac{y}{h^2}\right) = -\frac{z}{h^2} \quad &&\text{in } \Omega, \\ &\nabla y \cdot \vec{\eta} + \big(K - \alpha'(1)\big)y = 0 &&\text{on } \partial\Omega. \end{aligned} \tag{3.17}$$

Multiply the top equation in (3.17) by z and integrate to obtain

$$\int_\Omega z\nabla \cdot \frac{1}{h^2}\nabla y\, dx - \int_\Omega \left(\frac{\nabla^2 h}{h^3} - \frac{2|\nabla h|^2}{h^4}\right)yz\, dx - \lambda_0 \int_\Omega \frac{yz}{h^2}\, dx = -\int_\Omega \left(\frac{z}{h}\right)^2 dx. \tag{3.18}$$

It follows from the divergence theorem and the second equation in (3.17) that

$$\int_\Omega z\nabla \cdot \left(\frac{1}{h^2}\nabla y\right) dx = \int_\Omega y\nabla \cdot \left(\frac{1}{h^2}\nabla z\right) dx. \tag{3.19}$$

Substituting (3.19) into (3.18) and employing (3.11) shows that

$$\int_\Omega \left(\frac{z}{h}\right)^2 dx = 0,$$

a contradiction. Consequently, there can be no such y and (3.15) holds.

To determine that $\lambda_0 > 0$, first observe that if $G_\rho(\lambda, 0)z = 0$ for some $\lambda \in \mathbb{R}$ and $z \in C^{2,\gamma}(\bar{\Omega})$, then $\phi = z/h$ satisfies (3.3). Thus λ_0 may be characterized also as the unique real value for which (3.3) admits a positive solution. For purposes of estimating λ_0, one can work with either (3.3) or (3.11). Note that (3.3) gives

$$\lambda_0 = \frac{\nabla^2 \phi}{\phi}. \tag{3.20}$$

Since $\nabla \cdot (\frac{\nabla\phi}{\phi}) = \frac{\nabla^2\phi}{\phi} - \frac{|\nabla\phi|^2}{\phi^2}$, integrating (3.20) yields

$$\begin{aligned}
|\Omega|\lambda_0 &= \int_\Omega \frac{\nabla^2\phi}{\phi}\,dx \\
&= \int_\Omega \nabla\cdot\left(\frac{\nabla\phi}{\phi}\right)dx + \int_\Omega \frac{|\nabla\phi|^2}{\phi^2}\,dx \\
&= \int_{\partial\Omega} \frac{\nabla\phi\cdot\vec{\eta}}{\phi}\,dS + \int_\Omega \frac{|\nabla\phi|^2}{\phi^2}\,dx \\
&= \alpha'(1)|\partial\Omega| + \int_\Omega \frac{|\nabla\phi|^2}{\phi^2}\,dx,
\end{aligned}$$

allowing us to conclude that

$$\lambda_0 > \alpha'(1)\frac{|\partial\Omega|}{|\Omega|} \quad \text{if } \alpha'(1) > 0. \tag{3.21}$$

(Note that when $\alpha'(1) = 0$, it is immediate from (3.3) that $\lambda_0 = 0$.)

To obtain an upper bound on λ_0, let k be a normalized principal eigenfunction for

$$\begin{aligned}
-\nabla^2 k &= \lambda k && \text{in } \Omega, \\
\nabla k\cdot\vec{\eta} + \alpha'(1)k &= 0 && \text{on } \partial\Omega.
\end{aligned} \tag{3.22}$$

By (1.13), λ in (3.22) is given by $\lambda = \lambda^1_{\alpha^*}(\Omega)$, where

$$\alpha^* = \frac{1}{1+\alpha'(1)}. \tag{3.23}$$

Now let

$$z = k\phi \tag{3.24}$$

where $k > 0$ is as in (3.22) and ϕ is as in (3.3). Then preceding as after (3.4), we have that z in (3.24) satisfies

$$\begin{aligned}
&\nabla\cdot\frac{1}{k^2}\nabla z - \left(\frac{\nabla^2 k}{k^3} - \frac{2|\nabla k|^2}{k^4}\right)z - \lambda_0\frac{z}{k^2} = 0 && \text{in } \Omega, \\
&\nabla z\cdot\vec{\eta} = 0 && \text{on } \partial\Omega.
\end{aligned} \tag{3.25}$$

The variational characterization of (3.25) [7] tells us that

$$\begin{aligned}
\lambda_0 &= \sup_{z\in W^{1,2}(\Omega)} \frac{\int_\Omega \big[-\frac{|\nabla z|^2}{k^2} - \big(\frac{\nabla^2 k}{k} - \frac{2|\nabla k|^2}{k^2}\big)\frac{z^2}{k^2}\big]\,dx}{\int_\Omega \frac{z^2}{k^2}\,dx} \\
&\leqslant \sup_{z\in W^{1,2}(\Omega)} \frac{\int_\Omega \big(-\frac{\nabla^2 k}{k} + \frac{2|\nabla k|^2}{k^2}\big)\frac{z^2}{k^2}\,dx}{\int_\Omega \frac{z^2}{k^2}\,dx} \\
&\leqslant \sup_{z\in W^{1,2}(\Omega)} \frac{\int_\Omega \big(-\frac{\nabla^2 k}{k}\big) z^2/k^2\,dx}{\int_\Omega z^2/k^2\,dx} \\
&\quad + \sup_{z\in W^{1,2}(\Omega)} \frac{\int_\Omega 2|\nabla k|^2/k^2 \cdot \frac{z^2}{k^2}\,dx}{\int_\Omega z^2/k^2\,dx} \\
&\leqslant \lambda^1_{\alpha^*}(\bar\Omega) + \sup_{\bar\Omega} 2|\nabla k|^2/k^2.
\end{aligned}$$

Hence

$$\lambda_0 \leqslant \lambda^1_{\alpha^*}(\bar\Omega) + \sup_{\bar\Omega} 2|\nabla k|^2/k^2. \tag{3.26}$$

It follows from (3.22) and (3.23) that the right-hand side of (3.26) converges to 0 as $\alpha'(1) \to 0$. Moreover, the right-hand side of (3.21) can be made as large as desired simply by making $\alpha'(1)$ large enough. Consequently, not only is $\lambda_0 > 0$ if $\alpha'(1) > 0$, but it is also the case that as $\alpha'(1)$ ranges over the positive numbers, so does λ_0.

We now have that λ_0 is positive and that the hypotheses of the Crandall–Rabinowitz theorem are satisfied for $G(\lambda, \rho) = 0$. It follows from (3.4), (3.5), (3.6), (3.12), (3.13) and the definition of G that for (λ, ρ) in a neighborhood of $(\lambda_0, 0)$ and (λ, u) in a neighborhood of $(\lambda_0, 1)$, solutions to $G(\lambda, \rho) = 0$ correspond to solutions of $F(\lambda, u) = 0$, where F is as in (2.1), via the equivalence

$$G(\lambda, \rho) = 0 \quad \Leftrightarrow \quad F\left(\lambda, \frac{\rho + h}{h}\right) = 0. \tag{3.27}$$

By (3.27), we may return to (3.1) to determine the direction of bifurcation $\lambda'(s)$ $(s = 0)$ of the branch of equilibria to (1.1) which emanates from the line of solutions $(\lambda, 1)$ to (3.1) at the parameter value $\lambda = \lambda_0$. Our calculation of first and second derivatives with respect to s along this branch proceeds exactly as in (2.13) to (2.16). However, now when $s = 0$, $u \equiv 1$, $\lambda = \lambda_0$ and $u' = \phi$, where ϕ is as in (3.3). Since we are interested in equilibria to (1.1) with $u \leqslant 1$ on $\bar\Omega$, the branch of primary interest to us is that corresponding to $s < 0$ and small.

Substituting into (2.13)–(2.14) and (2.15)–(2.16) and calculating along the lines of (2.17)–(2.20), we obtain

$$\nabla^2\phi - \lambda_0\phi = 0 \quad \text{in } \Omega, \tag{3.28}$$

$$\nabla\phi \cdot \vec\eta - \frac{d\alpha}{du}(1)\phi = 0 \quad \text{on } \partial\Omega \tag{3.29}$$

and

$$\nabla^2 u'' - 2\lambda'(0)\phi - 2\lambda_0\phi^2 - \lambda_0 u'' = 0 \quad \text{in } \Omega, \tag{3.30}$$

$$2\frac{d\alpha}{du}(1)\phi\nabla\phi\cdot\vec{\eta}+\nabla u''\cdot\vec{\eta}-\frac{d^2\alpha}{du^2}(1)\phi^2-\frac{d\alpha}{du}(1)u''-2\frac{d\alpha}{du}(1)\phi^2=0 \quad \text{on } \partial\Omega. \tag{3.31}$$

To calculate $\lambda'(0)$ here, we multiply (3.28) by u'' and (3.30) by ϕ, integrate and employ Green's second identity and (3.28)–(3.31). On one hand, we get

$$\begin{aligned}\int_\Omega (\phi\nabla^2 u''-u''\nabla^2\phi)\,dx &= \int_\Omega \big[\phi\big(2\lambda'(0)\phi+2\lambda_0\phi^2+\lambda_0 u''\big)-u''(\lambda_0\phi)\big]\,dx\\ &= 2\int_\Omega \big[\lambda'(0)\phi^2+\lambda_0\phi^3\big]\,dx,\end{aligned}$$

while on the other we have

$$\begin{aligned}\int_\Omega (\phi\nabla^2 u''-u''\nabla^2\phi)\,dx &= \int_{\partial\Omega} (\phi\nabla u''\cdot\vec{\eta}-u''\nabla\phi\cdot\vec{\eta})\,dS\\ &= \int_{\partial\Omega}\bigg[\phi\bigg(-2\frac{d\alpha}{du}(1)\phi\nabla\phi\cdot\vec{\eta}+\frac{d^2\alpha}{du^2}(1)\phi^2\\ &\quad+\frac{d\alpha}{du}(1)u''+2\frac{d\alpha}{du}(1)\phi^2\bigg)-u''\bigg(\frac{d\alpha}{du}(1)\phi\bigg)\bigg]\,dS\\ &= \int_{\partial\Omega}\phi^3\bigg[-2\bigg(\frac{d\alpha}{du}(1)\bigg)^2+2\frac{d\alpha}{du}(1)+\frac{d^2\alpha}{du^2}(1)\bigg]\,dS.\end{aligned}$$

It now follows that

$$\lambda'(0)=\frac{\int_{\partial\Omega}\phi^3\big[\frac{d^2\alpha}{du^2}(1)+2\frac{d\alpha}{du}(1)-2\big(\frac{d\alpha}{du}(1)\big)^2\big]\,dS-2\lambda_0\int_\Omega\phi^3\,dx}{2\int_\Omega\phi^2\,dx}. \tag{3.32}$$

Note that in (3.32), λ_0 and ϕ depend on $\frac{d\alpha}{du}(1)$, but are independent of the choice of $\frac{d^2\alpha}{du^2}(1)$. Consequently, once any fixed value of $\frac{d\alpha}{du}(1)>0$ is specified (and hence λ_0 is determined), the direction of bifurcation $\lambda'(0)$ of the branch of solutions to (3.1) emanating from the line of solutions $(\lambda, 1)$ at λ_0 is controlled solely by the value of $\frac{d^2\alpha}{du^2}(1)$. Indeed, $\lambda'(0)$ may be positive, negative or zero. Recall that the branch of solutions $(\lambda(s), u(s))$ of interest to us corresponds to $s<0$ and small. So, for instance, when $\lambda'(0)>0$, $\lambda(s)<\lambda_0$ for such solutions for s near enough to 0.

We now want to determine the stability of the equilibrium solutions to (1.1) near the bifurcation point $(\lambda_0, 1)$ thought of as solutions to (1.1). To do so, we recast the time dependent problem (1.1) in terms of ρ, as before, obtaining

$$\begin{aligned} &\frac{\partial \rho}{\partial t} = \nabla^2 \rho - \frac{2\nabla h \cdot \nabla \rho}{h} - \left(\frac{\nabla^2 h}{h} - \frac{2|\nabla h|^2}{h^2}\right)\rho + \lambda\left(\rho - \frac{\rho^2 + 2\rho h}{h}\right) \quad \text{in } \Omega \times (0,\infty), \\ &\alpha\left(\frac{\rho}{h}+1\right)\nabla(\rho+h)\cdot\vec{\eta} + \left[(K-1)\alpha\left(\frac{\rho}{h}+1\right)+1\right](\rho+h) = 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{aligned} \tag{3.33}$$

The linearization about equilibria to (3.33) with respect to ρ at $\rho = 0$ yields the expressions

$$\begin{aligned} &\nabla^2 z - \frac{2\nabla h \cdot \nabla z}{h} - \left(\frac{\nabla^2 h}{h} - \frac{2|\nabla h|^2}{h^2}\right) z - \lambda z \quad \text{in } \Omega, \\ &\nabla z \cdot \vec{\eta} + \left(K - \alpha'(1)\right) z \quad \text{on } \partial\Omega. \end{aligned}$$

As in (2.24), we may determine the linearized stability of the zero equilibrium to (3.33) at $\lambda \geqslant 0$ via the sign of μ in

$$\begin{aligned} &\nabla^2 z - \frac{2\nabla h \cdot \nabla z}{h} - \left(\frac{\nabla^2 h}{h} - \frac{2|\nabla h|^2}{h^2}\right) z - \lambda z = \mu z \quad \text{in } \Omega, \\ &\nabla z \cdot \vec{\eta} + \left(K - \alpha'(1)\right) z = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.34}$$

where $z > 0$ on $\bar{\Omega}$. If one divides the top equation in (3.34) by h^2, then (3.34) may be recast as

$$G_\rho(\lambda + \mu, 0) z = 0. \tag{3.35}$$

Since $z > 0$ in $\bar{\Omega}$, it follows from (3.35) that $\lambda + \mu = \lambda_0$. So $\mu < 0$ if $\lambda > \lambda_0$ and $\mu > 0$ if $\lambda < \lambda_0$. Consequently, we may conclude that $u \equiv 1$ is asymptotically stable as an equilibrium to (1.1) when $\lambda > \lambda_0$ and unstable when $\lambda < \lambda_0$. The stability of the branch of equilibria $(\lambda(s), u(s))$ emanating from $(\lambda, 1)$ at $\lambda = \lambda_0$ and corresponding to $s < 0$ and small can now be assessed via the extension of the Crandall–Rabinowitz theorem [8] and we conclude that equilibria along the branch are asymptotically stable when $\lambda'(0) > 0$ and so $\lambda(s) < \lambda_0$ for small s and unstable when $\lambda'(0) < 0$ and hence $\lambda(s) > \lambda_0$ for small s.

We can combine the results of this section and the preceding one to draw some conclusions about Allee effects and multiple equilibria in (1.1) when $\alpha(0) > 0$, $\alpha(1) = 1$ and $\alpha'(1) > 0$. To this end, first recall that equilibria to (1.1) emanate from $(\lambda, 0)$ at $\lambda = \lambda^1_{\alpha(0)}(\Omega)$ and emanate from $(\lambda, 1)$ at $\lambda = \lambda_0$. The value of $\lambda^1_{\alpha(0)}(\Omega)$ is determined in (1.13) and depends solely and continuously on the value of $\alpha(0) \in (0, 1)$. Hence $\lambda^1_{\alpha(0)}(\Omega)$ may take on any value in the interval $(0, \lambda^1_0(\Omega))$. On the other hand, λ_0 is determined solely by $\alpha'(1)$ and it follows from (3.21) and (3.26) that λ_0 may assume any positive value. As a result, any one of the three possibilities $\lambda_0 < \lambda^1_{\alpha(0)}(\Omega)$, $\lambda_0 = \lambda^1_{\alpha(0)}(\Omega)$, $\lambda_0 > \lambda^1_{\alpha(0)}(\Omega)$ may obtain.

Let us suppose now that $\lambda_0 < \lambda^1_{\alpha(0)}(\Omega)$. Then we have that $u \equiv 0$ and $u \equiv 1$ are both asymptotically stable as equilibrium solutions to (1.1) for $\lambda \in (\lambda_0, \lambda^1_{\alpha(0)}(\Omega))$. So for such values of $\lambda = r/d$, solutions to (1.1) with $u(x, 0)$ positive and sufficiently small decay over time to 0 while solutions to (1.1) with $u(x, 0)$ less than but close to 1 on $\bar{\Omega}$ converge over time to 1. Clearly, in such a case, (1.1) exhibits what amounts to an Allee effect for (r, d) with $r/d \in (\lambda_0, \lambda^1_{\alpha(0)}(\Omega))$. Here the Allee effect is reminiscient of those we displayed in [5]. In [5], we presumed $\alpha(0)$ to be small while $\alpha(u) \in (0, 1)$ was relatively large for u close enough to 1. However, the value of $\alpha(0)$ could be relatively arbitrary in $(0, 1)$ and the construction of [5] would remain valid. The primary effect of increasing the value of $\alpha(0)$ would be to narrow the range of parameter values $\lambda = r/d$ for which the effect would be detected. Presently, we can have $\alpha(0)$ take on any value in $(0, 1)$, with $\lambda^1_{\alpha(0)}(\Omega)$ approaching 0 as $\alpha(0)$ approaches 1. So again, if $\alpha(0)$ is increased, the range of parameter values λ for which we can detect an Allee effect in (1.1) by virtue of having $\lambda \in (\lambda_0, \lambda^1_{\alpha(0)}(\Omega))$ is necessarily narrowed. At the other end, if $\alpha(1) = 1$ and $\alpha'(1) > 0$ but sufficiently small, $\alpha(u) \in (0, 1)$ will be relatively large when u is less than but close to 1.

Suppose next that $\lambda_0 = \lambda^1_{\alpha(0)}(\Omega)$. If the branch of equilibria to (1.1) given by $(\lambda(s), u(s))$, $-1 \ll s < 0$ which emanates from $(\lambda, 1)$ at λ_0 is such that $\lambda'(0) > 0$, then for $\lambda = \lambda(s) < \lambda_0$, $u(s)$ is a large asymptotically stable equilibrium to (1.1) and 0 is also asymptotically stable. Consequently, in this case, (1.1) exhibits an Allee effect. If $\lambda'(0) < 0$, (1.1) may exhibit multiple stable equilibria and hence a threshold effect along the lines of [12]. Additionally, if the branch of equilibria to (1.1) which emanates from $(\lambda, 0)$ at $\lambda^1_{\alpha(0)}(\Omega) = \lambda_0$ bends in the same direction with respect to λ as does $\{(\lambda(s), u(s))\}$, then (1.1) has at least three nontrivial equilibria (including $u \equiv 1$) for λ near λ_0 (here $\lambda > \lambda_0$ in the case wherein $\lambda'(0) < 0$ and $\lambda < \lambda_0$ when $\lambda'(0) > 0$).

When $\lambda_0 > \lambda^1_{\alpha(0)}(\Omega)$, notice that $u \equiv 0$ is unstable as a solution to (1.1) for λ near λ_0 and that $u \equiv 1$ is unstable when λ is near $\lambda^1_{\alpha(0)}(\Omega)$. Consequently, obtaining Allee effects on the basis of local bifurcation from $u \equiv 1$ as described above is no longer possible. We can still obtain Allee effects and multiple equilibria on the basis of subcritical bifurcation from the trivial equilibria at $\lambda = \lambda^1_{\alpha(0)}(\Omega)$, as in the preceding section. Our description of this phenomenon will be enhanced if we employ global as well as local bifurcation results. We establish the necessary results in the next section and continue this discussion there.

When $\alpha(0) = 0$, the results of [5] show that $u \equiv 0$ remains an asymptotically stable equilibrium to (1.1) when $\lambda < \lambda^1_{\alpha(0)}(\Omega) = \lambda^1_0(\Omega)$. Consequently, much of the preceding discussion remains valid if $\alpha(0) = 0$, $\alpha'(0) > 0$, $\alpha(1) = 1$ and $\alpha'(1) > 0$. Certainly, if $\lambda_0 < \lambda^1_0(\Omega)$, (1.1) exhibits an Allee effect for $\lambda \in (\lambda_0, \lambda^1_0(\Omega))$ just as before. When $\lambda_0 = \lambda^1_0(\Omega)$, we get an Allee effect when $\lambda'(0) > 0$ and multiple equilibria when $\lambda'(0) < 0$. However, if $\lambda_0 > \lambda^1_0(\Omega)$, the results of Section 2 show that we cannot obtain an Allee effect on the basis of subcritical bifurcation from the trivial equilibria at $\lambda = \lambda^1_0(\Omega)$.

4. Global bifurcation results

In order to establish global results about the set of equilibria to (1.1) which bifurcate from either the set of trivial equilibria $\{(\lambda, 0)\colon \lambda \in \mathbb{R}\}$ or the set $\{(\lambda, 1)\colon \lambda \in \mathbb{R}\}$, we again need to distinguish between the cases $\alpha(0) = 0$ and $\alpha(0) > 0$. Some of the reasons for this distinction are technical. However, the distinction is also related to the possibility of linking solutions that emanate from $u \equiv 0$ to solutions that emanate from $u \equiv 1$.

We will begin our discussion assuming that $\alpha(0) > 0$. In this situation, in Section 2, when we were concerned with local bifurcation from the set of trivial equilibria, we could formulate the question in terms of solutions to

$$F(\lambda, u) = 0 = (0, 0), \tag{4.1}$$

where $F(\lambda, u)$ is as in (2.1). Now, however, we can no longer assume that u is in a small neighborhood of 0 in $C^{2+\alpha}(\bar{\Omega})$. Consequently, we employ the positive auxiliary function h given in (3.12) (assuming (3.8) and (3.13)) so as to reformulate (4.1) along the lines (3.5)–(3.6); namely, we consider

$$\nabla \cdot \frac{1}{h^2}\nabla w - \left(\frac{\nabla^2 h}{h^3} - \frac{2|\nabla h|^2}{h^4}\right)w + \lambda\left(\frac{w}{h^2} - \frac{w^2}{h^3}\right) = 0 \tag{4.2}$$

in Ω with

$$\alpha\left(\frac{w}{h}\right)\nabla w \cdot \vec{\eta} - \left[\alpha\left(\frac{w}{h}\right)\frac{\nabla h \cdot \vec{\eta}}{h} - \left(1 - \alpha\left(\frac{w}{h}\right)\right)\right]w = 0 \tag{4.3}$$

on $\partial\Omega$. (Recall that since $\alpha(0) > 0$, we may assume that (1.4) holds and also that w in (4.2)–(4.3) corresponds to hu by (3.4).)

Now choose an $M > 0$ so that

$$\frac{\nabla^2 h}{h^3} - \frac{2|\nabla h|^2}{h^4} + \frac{M}{h^2} > 0 \tag{4.4}$$

on $\bar{\Omega}$ and rewrite (4.2) as

$$-\nabla \cdot \left(\frac{1}{h^2}\nabla w\right) + \left(\frac{\nabla^2 h}{h^3} - \frac{2|\nabla h|^2}{h^4} + \frac{M}{h^2}\right)w = (\lambda + M)\left(\frac{w}{h^2}\right) - \lambda\left(\frac{w^2}{h^3}\right) \tag{4.5}$$

in Ω. Then for any fixed $w \in C^{1,\gamma}(\bar{\Omega})$, it follows from (4.4), (3.8) and (3.13) that the equations

$$-\nabla \cdot \left(\frac{1}{h^2}\nabla z\right) + \left(\frac{\nabla^2 h}{h^3} - \frac{2|\nabla h|^2}{h^4} + \frac{M}{h^2}\right)z = y \quad \text{in } \Omega, \tag{4.6}$$

$$\alpha\left(\frac{w}{h}\right)\nabla z \cdot \vec{\eta} - \left[\alpha\left(\frac{w}{h}\right)\frac{\nabla h \cdot \vec{\eta}}{h} - \left(1 - \alpha\left(\frac{w}{h}\right)\right)\right]z = p \quad \text{on } \partial\Omega \tag{4.7}$$

define a continuous linear solution operator from $C^{\gamma}(\bar{\Omega}) \times C^{1,\gamma}(\partial\Omega)$ into $C^{2,\gamma}(\bar{\Omega})$. Let us denote this operator by $\mathcal{A}(w)$, so that (4.6)–(4.7) can be expressed as

$$z = \mathcal{A}(w)(y, p). \tag{4.8}$$

By [9], we can view $\mathcal{A}(w)$ as a compact operator from $C^{1,\gamma}(\bar{\Omega}) \times C^{1,\gamma}(\partial\Omega)$ into $C^{1,\gamma}(\bar{\Omega})$.

So now define a continuous linear operator $A(w) : C^{\gamma}(\bar{\Omega}) \to C^{2,\gamma}(\bar{\Omega})$ by

$$A(w)y = \mathcal{A}(w)(y, 0). \tag{4.9}$$

(We may view $A(w)$ as a compact map from $C^{1,\gamma}(\bar{\Omega})$ to $C^{1,\gamma}(\bar{\Omega})$.) Then (4.5) and (4.3) are equivalent to the operator equation

$$w = A(w)\left((\lambda + M)\left(\frac{w}{h^2}\right) - \lambda\left(\frac{w^2}{h^3}\right)\right). \tag{4.10}$$

It follows from [6, Theorem 1.2] and [11, Chapter 3, Theorem 3.1] that if $\{\lambda_k\}$ is bounded in $\mathbb{R}$ and $\{w_k\}$ is bounded in $C^{1,\gamma}(\bar{\Omega})$, then

$$A(w_k)\left((\lambda_k + M)\left(\frac{w_k}{h^2}\right) - \lambda_k\left(\frac{w_k^2}{h^3}\right)\right)$$

is bounded in $C^{2,\gamma}(\bar{\Omega})$ and hence is precompact in $C^{1,\gamma}(\bar{\Omega})$. That the right-hand side of (4.10) is a differentiable map from $C^{1,\gamma}(\bar{\Omega})$ to $C^{1,\gamma}(\bar{\Omega})$ is also a consequence of [11, Chapter 3, Theorem 3.1]. To see that such is the case, we proceed as follows. Notice first that if $p = 0$ in (4.7), (4.7) can be written as

$$\nabla z \cdot \vec{\eta} + \left(K - 1 + \frac{1}{\alpha(\frac{w}{h})}\right)z = 0 \quad \text{on } \partial\Omega. \tag{4.11}$$

(Here we use (3.13).) Then for $\rho \in C^{1,\gamma}(\bar{\Omega})$, define the compact linear operator $\mathcal{B}(\rho): C^{\gamma}(\bar{\Omega}) \times C^{1,\gamma}(\partial\Omega) \to C^{1,\gamma}(\bar{\Omega})$ by

$$z = \mathcal{B}(\rho)(y, p)$$

if and only if

$$-\nabla \cdot \left(\frac{1}{h^2}\nabla z\right) + \left(\frac{\nabla h}{h^2} - \frac{2|\nabla h|^2}{h^4} + \frac{M}{h^2}\right)z = y \quad \text{in } \Omega, \tag{4.12}$$

$$\nabla z \cdot \vec{\eta} + (K - 1 + \rho)z = p \quad \text{on } \partial\Omega. \tag{4.13}$$

So by (4.11), the right-hand side of (4.10) is differentiable in w so long as the map

$$\rho \to \mathcal{B}(\rho)(y, 0)$$

is differentiable in ρ as a map from $C^{1,\gamma}(\bar{\Omega})$ to $C^{1,\gamma}(\bar{\Omega})$. Let $y \in C^{1,\gamma}(\bar{\Omega})$ be fixed and let $z_1 = \mathcal{B}(\rho)(y, 0)$ and $z_2 = \mathcal{B}(\rho + v)(y, 0)$, where $v \in C^{1,\gamma}(\bar{\Omega})$. By (4.12)–(4.13),

$$\mathcal{B}(\rho + v)(y, 0) - \mathcal{B}(\rho)(y, 0) = z_2 - z_1 = \mathcal{B}(\rho)(0, -z_2 v). \tag{4.14}$$

Since Theorem 3.1 of [11, Chapter 3] guarantees that $z_2 \in C^{2,\gamma}(\bar{\Omega})$ is bounded if $v \in C^{1,\gamma}(\bar{\Omega})$ is bounded, it follows from (4.14) that $\mathcal{B}(\rho)(y, 0)$ is continuous in ρ. Consequently,

$$\begin{aligned} &\mathcal{B}(\rho + v)(y, 0) - \mathcal{B}(\rho)(y, 0) - \mathcal{B}(\rho)\big(0, -v\mathcal{B}(\rho)(y, 0)\big) \\ &\quad = \mathcal{B}(\rho)(0, -z_2 v) - \mathcal{B}(\rho)(0, -z_1 v) = \mathcal{B}(\rho)\big(0, -(z_2 - z_1)v\big), \end{aligned}$$

which is $o(\|v\|)$ since $z_2 \to z_1$ as $v \to 0$ by the continuity of $\mathcal{B}(\rho)(y, 0)$ in ρ. Hence $\mathcal{B}(\rho)(y, 0)$ is differentiable in ρ and thus the right-hand side of (4.10) is a differentiable map from $C^{1,\gamma}(\bar{\Omega})$ to $C^{1,\gamma}(\bar{\Omega})$. As a consequence, a direct calculation shows that we can recast (4.10) as

$$w = A(0)(\lambda + M)\left(\frac{w}{h^2}\right) + R(\lambda, w), \tag{4.15}$$

where $R(\lambda, w) = o(\|w\|)$ uniformly for λ in bounded subsets of $\mathbb{R}$. So we have an appropriate functional analytic setting in which to apply the Rabinowitz global bifurcation theorem [6,14] to the set of equilibria to (1.1) which bifurcate from the set of trivial solutions $\{(\lambda, 0){:}\ \lambda \in \mathbb{R}\}$ under the assumption that $\alpha(0) > 0$.

Keeping the assumption that $\alpha(0) > 0$, it is straight forward to modify the preceding discussion by substituting

$$w = \rho + h \tag{4.16}$$

into (4.5) and (4.3) to establish that the set of equilibria to (1.1) which emanate from $\{(\lambda, 1){:}\ \lambda \in \mathbb{R}\}$ can also be analyzed via the Rabinowitz global bifurcation theorem. We leave the details to the interested reader.

Let us now turn to the case $\alpha(0) = 0$, where we also assume $\alpha'(0) > 0$. In this case, we established in Section 2 that the positive equilibria to (1.1) which bifurcate from the set of trivial solutions must themselves vanish on the boundary $\partial\Omega$ of the domain Ω, and noted there the global description of this set as the smooth arc $\{(\lambda, \bar{u}(\lambda)){:}\ \lambda > \lambda_0^1(\Omega)\}$ in $\mathbb{R} \times C^{2,\gamma}(\bar{\Omega})$. Furthermore, assuming Ω is not a ball in $\mathbb{R}^N$, we also showed in Section 2 that $\{(\lambda, \bar{u}(\lambda)){:}\ \lambda > \lambda_0^1(\Omega)\}$ is isolated in a reasonable sense as a subset of the positive equilibria to (1.1). For the present, we will assume this form of isolation for $\{(\lambda, \bar{u}(\lambda)){:}\ \lambda > \lambda_0^1(\Omega)\}$. Under this assumption, our discussion of global bifurcation of positive equilibria of (1.1) from the trivial solutions is complete in this case and we need only focus on establishing a suitable context for a global analysis of the equilibria to (1.1) which bifurcate from the set $\{(\lambda, 1){:}\ \lambda \in \mathbb{R}\}$. To this end, it follows from Section 2 that equilibria to (1.1) near $(\lambda^*, 1)$ for some $\lambda^* \geqslant 0$ must satisfy

$$\begin{aligned} &\nabla^2 u + \lambda u(1-u) = 0 && \text{in } \Omega, \\ &\beta(u)\nabla u \cdot \vec{\eta} + \big(1 - \alpha(u)\big) = 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.17}$$

where β is given by

$$\beta(u) = \begin{cases} \frac{\alpha(u)}{u}, & u \neq 0, \\ \alpha'(0), & u = 0 \end{cases}$$

and we may assume

$$\beta(\mathbb{R}) \subseteq (\delta, R) \tag{4.18}$$

for some $0 < \delta < R$. Letting

$$u = 1 + \rho \tag{4.19}$$

we can recast (4.17) as

$$\begin{aligned} &-\nabla^2 \rho = -\lambda\rho(1+\rho) && \text{in } \Omega, \\ &\nabla\rho \cdot \vec{\eta} + \rho = \frac{\alpha(1+\rho) - 1}{\beta(1+\rho)} + \rho && \text{on } \partial\Omega. \end{aligned} \tag{4.20}$$

Now let $L : C^{\gamma}(\bar{\Omega}) \times C^{1,\gamma}(\partial\Omega) \to C^{2,\gamma}(\bar{\Omega})$ be the bounded linear operator given by

$$z = L(f, g) \tag{4.21}$$

if and only if

$$\begin{aligned} -\nabla^2 z &= f && \text{in } \Omega, \\ \nabla z \cdot \vec{\eta} + z &= g && \text{on } \partial\Omega. \end{aligned} \tag{4.22}$$

By (4.21)–(4.22), (4.20) is equivalent to

$$\rho = L\left(-\lambda\rho(1+\rho), \frac{\alpha(1+\rho)-1}{\beta(1+\rho)} + \rho\right), \tag{4.23}$$

which is of the form

$$\rho = \mathcal{F}(\lambda, \rho), \tag{4.24}$$

where $\mathcal{F}$ is a map from $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ into $C^{1,\gamma}(\bar{\Omega})$. It follows from (4.18) and (4.20)–(4.21) that $\mathcal{F}$ is compact and continuous, uniformly for λ in bounded intervals.

In this case, the fact that λ occurs only in the first component of the right-hand side of (4.23) means that (4.24) is not of the form to which the original Rabinowitz global bifurcation theorem [14] applies, in contrast to (4.15). Of course, there are various extensions of Rabinowitz's work that will apply to (4.24), among them that of Alexander and Antman [1], which we will employ in the case of (4.24).

Now, whether we are considering (4.15) or (4.24) (and consequently using Rabinowitz's original results or Alexander and Antman's extension), the essential element that must be verified in order to make assertions about the global dispensation of bifurcating continua of equilibria to (1.1) is the same. Namely, we need to verify that for appropriate values of λ the generalized null space of the linearization about 0 is of odd dimension. In the cases under consideration the dimension is one; i.e. we have algebraic simplicity. In the case of (4.15) the necessary argument is rather straight forward and we leave it to the interested reader to verify the result. However, for (4.24) the formulation is a bit more novel, so we have chosen to include it here. It follows from (4.21)–(4.23) that the relevant value of λ is λ_0, where λ_0 is given in (3.14) and that if

$$z = \mathcal{F}_{\rho}(\lambda_0, z), \tag{4.25}$$

then

$$z = L\big(-\lambda_0 z, \big(\alpha'(1) + 1\big)z\big)$$

so that

$$\begin{aligned} -\nabla^2 z &= -\lambda_0 z && \text{in } \Omega, \\ \nabla z \cdot \vec{\eta} - \alpha'(1) z &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.26}$$

which is just (3.3). Hence $\dim N(I - \mathcal{F}_\rho(\lambda_0, \cdot)) = 1$. Let $N(I - \mathcal{F}_\rho(\lambda_0, \cdot)) = \langle \phi \rangle$, where $\phi > 0$ on $\bar{\Omega}$. Then if $w \in N((I - \mathcal{F}_\rho(\lambda_0, \cdot))^2)$,

$$w - L\big(-\lambda_0 w, \big(\alpha'(1) + 1\big)w\big) = c\phi \tag{4.27}$$

for some $c \in \mathbb{R}$. Then

$$w - c\phi = L\big(-\lambda_0 w, \big(\alpha'(1) + 1\big)w\big)$$

and (4.21)–(4.23) implies

$$\begin{aligned} -\nabla^2(w - c\phi) &= -\lambda_0 w && \text{in } \Omega, \\ \nabla(w - c\phi) \cdot \vec{\eta} + (w - c\phi) &= \big(\alpha'(1) + 1\big)w && \text{on } \partial\Omega. \end{aligned} \tag{4.28}$$

Now multiply (4.28) by ϕ and integrate, obtaining

$$\int_\Omega -\phi \nabla^2 w \, dx + c \int_\Omega \phi \nabla^2 \phi \, dx = -\lambda_0 \int_\Omega \phi w \, dx. \tag{4.29}$$

The first term on the left-hand side of (4.29) becomes

$$\begin{aligned} \int_\Omega -\phi \nabla^2 w \, dx &= \int_\Omega -w \nabla^2 \phi \, dx + \int_{\partial\Omega} (-\phi \nabla w \cdot \vec{\eta} + w \nabla \phi \cdot \vec{\eta}) \, dS \\ &= -\lambda_0 \int_\Omega w\phi \, dx + \int_{\partial\Omega} \big(-\phi c \nabla \phi \cdot \vec{\eta} + c\phi + \alpha'(1) w\big) \, dS \\ &\quad + \int_{\partial\Omega} w\big(\alpha'(1)\phi\big) \, dS \\ &= -\lambda_0 \int_\Omega w\phi \, dx - c \int_{\partial\Omega} \big(\alpha'(1) + 1\big)\phi^2 \, dS. \end{aligned} \tag{4.30}$$

Since $\phi \nabla^2 \phi + |\nabla \phi|^2 = \operatorname{div}(\phi \nabla \phi)$, the second term on the left-hand side of (4.29) yields

$$\int_\Omega \phi \nabla^2 \phi = -\int_\Omega |\nabla \phi|^2 \, dx + \alpha'(1) \int_{\partial\Omega} \phi^2 \, dS. \tag{4.31}$$

Substituting (4.30) and (4.31) into (4.29) and simplifying we obtain

$$c\bigg[\int_{\partial\Omega} \phi^2 \, dS + \int_\Omega |\nabla \phi|^2 \, dx\bigg] = 0, \tag{4.32}$$

whence we conclude that $c = 0$ and hence that

$$N\big((I = \mathcal{F}_\rho(\lambda_0, \cdot))^2\big) = N\big(I - \mathcal{F}_\rho(\lambda_0, \cdot)\big),$$

as required.

We have now established the following result.

Theorem 4.1. *Consider the equilibrium solutions to* (1.1) *where $\alpha(u)$ is smooth, nondecreasing and satisfies* (1.2) *with $\alpha(1) = 1$ and $\alpha'(1) > 0$.*

(i) *If $\alpha(0) > 0$, assume* (1.4). *Let $h \in C^{2,\gamma}(\bar{\Omega})$ be a positive function and K a positive constant so that* (3.8) *and* (3.13) *hold and let M be a positive constant so that* (4.4) *holds.*
(ii) *If $\alpha(0) = 0$, assume $\alpha'(0) > 0$ and that* (4.18) *holds. Assume also that positive solutions to*

$$\begin{aligned} &\nabla^2 u + \lambda u(1-u) = 0 \quad \text{in } \Omega, \\ &u = 0 \qquad\qquad\qquad\quad\;\; \text{on } \partial\Omega \end{aligned} \tag{4.33}$$

are isolated among the equilibrium solutions to (1.1) *in the sense described in Section* 2. *Then:*

(a) *There exists a continuum of equilibrium solutions (λ, u) to* (1.1) *with $0 \leqslant u \leqslant 1$ on $\bar{\Omega}$ which emanates from $\{(\lambda, 0)\colon \lambda \in \mathbb{R}\}$ at $\lambda = \lambda^1_{\alpha(0)}(\Omega)$, where $\lambda^1_{\alpha(0)}(\Omega) > 0$ is as given in* (1.13) *and which is unbounded in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$.*
(b) *There exists a continuum of equilibrium solutions (λ, u) to* (1.1) *with $0 \leqslant u \leqslant 1$ on $\bar{\Omega}$ which emanates from $\{(\lambda, 1)\colon \lambda \in \mathbb{R}\}$ at $\lambda = \lambda_0$, where $\lambda_0 > 0$ is as given in* (3.14) *and which is unbounded in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$.*

Remark. The unboundedness of the continua in Theorem 4.1(a) and (b) is argued as follows. The exposition preceding the statement of the theorem enables us to apply Rabinowitz's Global Bifurcation Theorem (or one of its generalizations) to assert the existence of continua emanating from $\{(\lambda, 0)\colon \lambda \in \mathbb{R}\}$ and $\{(\lambda, 1)\colon \lambda \in \mathbb{R}\}$. Each of these continua must then satisfy Rabinowitz's global alternatives relative to its base (or "trivial") set of solutions. Since $\lambda^1_{\alpha(0)}(\Omega)$ and λ_0 are the unique parameter values from which solutions (λ, u) with $0 < u < 1$ on Ω may emanate from $\{(\lambda, 0)\colon \lambda \in \mathbb{R}\}$ and $\{(\lambda, 1)\colon \lambda \in \mathbb{R}\}$, respectively, it must be the case that both continua are unbounded in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$.

The preceding observation can be refined substantially. Denote the continuum emanating from $\{(\lambda, 0)\colon \lambda \in \mathbb{R}\}$ by $\mathcal{C}_0$ and the continuum emanating from $\{(\lambda, 1)\colon \lambda \in \mathbb{R}\}$ by $\mathcal{C}_1$. Since $0 \leqslant u \leqslant 1$ for points $(\lambda, u) \in \mathcal{C}_i$ for $i = 0, 1$, the assumptions (1.4) and (4.18) on α allow us employ regularity theory to conclude that for $i = 0$ or 1

$$\big\{\|u\|_{C^{2,\gamma}(\bar{\Omega})}\colon (\lambda, u) \in \mathcal{C}_i \text{ and } \lambda \in [a, b]\big\}$$

is bounded for any finite bounded interval $[a, b]$. Consequently, the only way in which $\mathcal{C}_i$, $i = 0, 1$ can be unbounded is to have $\{\lambda \in \mathbb{R}\colon (\lambda, u) \in \mathcal{C}_i\}$ be unbounded.

As previously noted, only equilibria (λ, u) with $\lambda \geqslant 0$ have any relevance to the underlying application. It is indeed the case that for $i = 0, 1$ the set $\pi(\mathcal{C}_i) = \{\lambda \in \mathbb{R}\colon (\lambda, u) \in \mathcal{C}_i \text{ for some } u \in C^{1,\gamma}(\bar{\Omega}) \text{ with } 0 < u < 1 \text{ on } \Omega\}$ is contained in $(0, \infty)$. Since $\lambda^1_{\alpha(0)}(\Omega)$ and λ_0 are both positive, this fact is a direct corollary of the following result.

Proposition 4.2. *Assume α is as in Theorem* 4.1. *Suppose that u satisfies*

$$\begin{aligned} &\nabla^2 u = 0 && \text{in } \Omega, \\ &\alpha(u)\nabla u \cdot \vec{\eta} + \big(1 - \alpha(u)\big)u = 0 && \text{on } \partial\Omega, \\ &0 \leqslant u \leqslant 1 && \text{on } \bar{\Omega}. \end{aligned} \tag{4.34}$$

Then $u \equiv 0$ or $u \equiv 1$.

Proof. We have $u\nabla^2 u = 0$ on Ω. Hence

$$\begin{aligned} 0 &= \int_\Omega u\nabla^2 u\,dx = \int_\Omega \nabla \cdot (u\nabla u)\,dx - \int_\Omega |\nabla u|^2\,dx \\ &= \int_{\partial\Omega} u\nabla u \cdot \vec{\eta}\,dS - \int_\Omega |\nabla u|^2\,dx = \int_\Gamma \left[\frac{\alpha(u) - 1}{\alpha(u)}\right] u^2\,dS - \int_\Omega |\nabla u|^2\,dx, \end{aligned}$$

where $\Gamma = \{x \in \partial\Omega\colon \alpha(u(x)) \neq 0\}$ and the integral over Γ is understood to be 0 if $\Gamma = \emptyset$. (If $\Gamma = \emptyset$, it follows from (4.34) that $u = 0$ on $\partial\Omega$.) Since $0 \leqslant u \leqslant 1$ on $\partial\Omega$ by (4.34), $0 \leqslant \alpha(u) \leqslant 1$ on $\partial\Omega$. Thus $\int_\Gamma [\frac{\alpha(u)-1}{\alpha(u)}]u^2\,dS \leqslant 0$, and so $\int_\Gamma [\frac{\alpha(u)-1}{\alpha(u)}]u^2\,dS = \int_\Omega |\nabla u|^2\,dx = 0$. Since $\int_\Omega |\nabla u|^2\,dx = 0$, u is constant on $\bar{\Omega}$. If $\Gamma = \emptyset$, clearly this constant must be 0. If $\Gamma \neq \emptyset$, then $\Gamma = \partial\Omega$ and if $u \equiv k$, then $\alpha(k) > 0$. It follows that

$$\alpha(k) = 1$$

and thus $k = 1$ by the assumptions on α.

Notice that it is completely consistent with global bifurcation theory to have $\pi(\mathcal{C}_i)$ bounded for $i = 0, 1$, so long as

$$\mathcal{C}_0 \cup \big(\mathbb{R} \times \{0\}\big) = \mathcal{C}_1 \cup \big(\mathbb{R} \times \{1\}\big). \tag{4.35}$$

In such a case, equilibrium solutions to (1.1) which emanate from $\mathbb{R} \times \{0\}$ in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ at $(\lambda^1_{\alpha(0)}(\Omega), 0)$ reach infinity by linking up to the ray of equilibrium solutions $\mathbb{R} \times \{1\}$ at $(\lambda_0, 1)$. So if (4.35) holds, $\pi(\mathcal{C}_0) = \pi(\mathcal{C}_1)$. When $\alpha(0) = 0$, the assumptions in Theorem 4.1 rule this possibility out. In that event, $\pi(\mathcal{C}_0) = (\lambda^1_{\alpha(0)}(\Omega), \infty)$ and $(\lambda_0, \infty) \subseteq \pi(\mathcal{C}_1)$. As a result, for all $\lambda > \max\{\lambda^1_{\alpha(0)}(\Omega), \lambda_0\}$, (1.1) has at least three equilibrium solutions (λ, u) with $0 \leqslant u \leqslant 1$ on $\bar{\Omega}$ and $u > 0$ on Ω (one of which is $u \equiv 1$). (For these values of λ, $u \equiv 1$ and $u \equiv 0$ are stable and unstable, respectively, as equilibria to (1.1). Since the nonzero equilibria to (1.1) emanating from $(\lambda^1_0(\Omega), 0)$ are stable as equilibria to (1.1) with $\alpha(u) \equiv 0$, one would expect a net instability among the equilibria to (1.1) which emanate from $(\lambda_0, 1)$.) When $\alpha(0) > 0$, if (4.35) does not hold, (1.1) will again have at least three nonzero equilibria for $\lambda > \max\{\lambda^1_{\alpha(0)}(\Omega), \lambda_0\}$. It is an extremely interesting question to ask whether there are choices of α with $\alpha(0) > 0$ for which $\pi(\mathcal{C}_i)$, $i = 0, 1$, are bounded. The answer, as we show next, is yes.

To establish that $\pi(\mathcal{C}_i)$, $i = 0, 1$, are bounded for some α with $\alpha(0) > 0$, we need to show that for such an α the set $\{u \in C^{1,\gamma}(\bar{\Omega})\colon (\lambda, u)$ is an equilibrium solution of (1.1) with $0 < u < 1$

on $\Omega\}$ is empty for all large enough values of λ. To this end, suppose for an α satisfying the conditions of Theorem 4.1 with $\alpha(0) > 0$ that for all $\lambda > \lambda^1_{\alpha(0)}(\Omega)$ there is a function u with

$$\begin{aligned} &\nabla^2 u + \lambda u(1-u) = 0 && \text{in } \Omega,\\ &\alpha(u)\nabla u \cdot \vec{\eta} + \big(1-\alpha(u)\big)u = 0 && \text{on } \partial\Omega,\\ &0 < u < 1 && \text{in } \Omega. \end{aligned} \tag{4.36}$$

Let λ_0 be as in (3.14) and let $\phi > 0$ satisfy

$$\begin{aligned} &\nabla^2 \phi - \lambda_0 \phi = 0 && \text{in } \Omega,\\ &\nabla\phi \cdot \vec{\eta} - \alpha'(1)\phi = 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.37}$$

and define I by

$$I = \int_\Omega \big[u\nabla^2\phi - \phi\nabla^2 u - \nabla^2\phi\big]\,dx. \tag{4.38}$$

It is straightforward to calculate on one hand that I in (4.38) equals

$$\begin{aligned} &\int_\Omega \big(\lambda_0 u\phi + \lambda\phi u(1-u) - \lambda_0\phi\big)\,dx\\ &\quad = \int_\Omega \big(\lambda\phi u(1-u) - \lambda_0\phi(1-u)\big)\,dx = \int_\Omega \lambda\phi(1-u)\left(u - \frac{\lambda_0}{\lambda}\right)dx. \end{aligned}$$

Now for any fixed α with $\alpha(0) > 0$ any positive solution to (4.36) is an upper solution to

$$\begin{aligned} &\nabla^2 u + \lambda u(1-u) = 0 && \text{in } \Omega,\\ &\alpha(0)\nabla u \cdot \vec{\eta} + \big(1-\alpha(0)\big)u = 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.39}$$

From (4.39) and the method of upper and lower solutions we conclude that $u \geqslant u_{\alpha(0)}(\lambda)$, the unique positive solution of (4.39). Hence

$$u - \frac{\lambda_0}{\lambda} \geqslant u_{\alpha(0)}(\lambda) - \frac{\lambda_0}{\lambda}. \tag{4.40}$$

It is immediate from the method of upper and lower solutions that $u_{\alpha(0)}(\lambda)$ increases in λ. Consequently, it follows from (4.40) that $I > 0$ for all large values of λ.

On the other hand, we have that

$$\begin{aligned} I &= \int_{\partial\Omega} [u\nabla\phi \cdot \vec{\eta} - \phi\nabla u \cdot \vec{\eta} - \nabla\phi \cdot \vec{\eta}]\,dS\\ &= \int_{\partial\Omega} \left[\alpha'(1)u\phi - \phi\left(1 - \frac{1}{\alpha(u)}\right)u - \alpha'(1)\phi\right]dS \end{aligned}$$

$$= \int_{\partial\Omega} \left(\alpha'(1)(u-1)\phi + \phi\left(\frac{1}{\alpha(u)} - 1\right)u \right) dS. \tag{4.41}$$

Now

$$\begin{aligned} \frac{1}{\alpha(u)} &= \frac{1}{1} - \frac{\alpha'(1)}{[\alpha(1)]^2}(u-1) + \frac{1}{2}\left[\frac{1}{\alpha(u)}\right]''\Big|_{u=\hat{u}}(u-1)^2 \\ &= 1 - \alpha'(1)(u-1) + \frac{1}{2}\left[\frac{1}{\alpha(u)}\right]''\Big|_{u=\hat{u}}(u-1)^2 \end{aligned}$$

where $u < \hat{u} < 1$. Substituting into (4.41) we have that

$$\begin{aligned} I &= \int_{\partial\Omega} \left(\alpha'(1)(u-1)\phi - \alpha'(1)(u-1)\phi u + \phi u\left(\frac{1}{2}\right)\left[\frac{1}{\alpha(u)}\right]''\Big|_{u=\hat{u}}(u-1)^2 \right) dS \\ &= \int_{\partial\Omega} \left(-\alpha'(1)\phi(u-1)^2 + \frac{1}{2}\left[\frac{1}{\alpha(u)}\right]''\Big|_{u=\hat{u}}(u-1)^2\phi u \right) dS \\ &= \int_{\partial\Omega} \phi(u-1)^2\left(-\alpha'(1) + \frac{1}{2}\left[\frac{1}{\alpha(u)}\right]''\Big|_{u=\hat{u}} u \right) dS \\ &\leqslant \int_{\partial\Omega} \phi(u-1)^2\left(-\alpha'(1) + \frac{u}{2}\max_{u\in[0,1]}\left[\frac{1}{\alpha(u)}\right]'' \right) dS. \end{aligned}$$

If α is such that

$$-\alpha'(1) + \frac{u}{2}\max_{u\in[0,1]}\left[\frac{2[\alpha'(u)]^2 - \alpha(u)\alpha''(u)}{[\alpha(u)]^3}\right] < 0 \quad \text{on } \partial\Omega \tag{4.42}$$

for any solution u of (4.36), then $I < 0$ for any $\lambda > \lambda^1_{\alpha(0)}(\Omega)$ and any corresponding u, contradicting the positivity of I for all large values of λ. So if (4.42) holds for some α with $\alpha(0) > 0$, $\pi(C_0)$ and $\pi(C_1)$ are bounded for that α.

It is easy to demonstrate that there are α satisfying the hypotheses of Theorem 4.1 with $\alpha(0) > 0$ which are also such that (4.42) holds. For example, let us assume that $\alpha''(u) > 0$. Then a very crude approximation tells us that

$$\begin{aligned} &-\alpha'(1) + \frac{u}{2}\max_{u\in[0,1]}\left[\frac{2[\alpha'(u)]^2 - \alpha(u)\alpha''(u)}{[\alpha(u)]^3}\right] \\ &\quad < -\alpha'(1) + \frac{[\alpha'(1)]^2}{[\alpha(0)]^3} = \alpha'(1)\left[-1 + \frac{\alpha'(1)}{[\alpha(0)]^3}\right]. \end{aligned}$$

Consequently, when $\alpha''(u) > 0$, (4.42) holds if

$$\alpha'(1) < [\alpha(0)]^3. \tag{4.43}$$

A standing assumption in our discussion is that $\alpha(0) \in (0, 1)$. So the condition (4.43) that we give for a concave α to be such that $\pi(\mathcal{C}_0)$ and $\pi(\mathcal{C}_1)$ are bounded requires that $\alpha'(1) < 1$, so that such an α is relatively flat. Of course, (4.43) is only a sufficient condition. It is an open question whether (4.35) always holds when α satisfies the conditions of Theorem 4.1 with $\alpha(0) > 0$. We should note that we can also give a somewhat constructive description of a function α meeting the conditions of Theorem 4.1 with $\alpha(0) > 0$ for which (4.35) holds, based on knowledge of solutions to (4.39) as $\alpha(0)$ approaches 1 and the method of upper and lower solutions. The function α which arises in the construction is again relatively flat on $[0, 1]$, although it does not need to satisfy $\alpha'(1) < 1$.

More generally, it is very natural to ask just what conditions on $\alpha(u)$ beyond having $\alpha(0) > 0$, $\alpha(1) = 1$ and $\alpha'(1) > 0$ are needed in order for (4.35) to hold. As noted, at present, we do not know the answer to this question. However, prompted by some interesting and useful feedback from the anonymous referee of this paper, we do have some thoughts on the subject that may be of some use in addressing the question. First of all, obviously it must be the case that $\alpha(1) = 1$ in order for $\{(\lambda, 1)\colon \lambda \in \mathbb{R}\}$ to be equilibria to (1.1) at all. So without the assumption $\alpha(1) = 1$ the question is meaningless. However, it seems to us that the fact that $\alpha(1) = 1$ will play more than just a basic background role in any successful resolution of the question. Let us elaborate. Notice that should (4.35) hold for some choice of $\alpha(u)$, the set $\mathcal{D} = \{u \in C^{1,\gamma}(\bar{\Omega})\colon 0 < u < 1$ on Ω and $(\lambda, u) \in \mathcal{C}_0$ for some $\lambda \in (0, \infty)\}$ is bounded and thus pre-compact in $C^1(\bar{\Omega})$. Indeed, it would be the case that $\{(\lambda, u) \in \mathbb{R} \times C^{1,\gamma}(\bar{\Omega})\colon 0 < u < 1$ on and $(\lambda, u) \in \mathcal{C}_0\}$ is bounded, but let us focus on $\mathcal{D}$.

Let $\alpha(u)$ satisfy (1.2) with $\alpha(0) > 0$ be fixed. Then if $(\lambda, u) \in \mathcal{C}_0$ with $0 < u < 1$ for $\lambda > \lambda^1_{\alpha(0)}(\Omega)$, the method of upper and lower solutions guarantees that $u > u_{\alpha(0)}(\lambda)$, the unique positive solution to

$$\begin{aligned} -\nabla^2 u &= \lambda u(1-u) && \text{in } \Omega, \\ \alpha(0)\nabla u \cdot \vec{\eta} + \big(1-\alpha(0)\big)u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We know that $\mathcal{C}_0$ for the equilibria to (1.1) when $\alpha(u) \equiv \alpha(0)$ is $\{(\lambda, u_{\alpha(0)}(\lambda))\colon \lambda > \lambda^1_{\alpha(0)}(\Omega)\}$ and that $u_{\alpha(0)}(\lambda)(x) \nearrow 1$ for $x \in \Omega$ as $\lambda \to \infty$. It follows that for our fixed choice of $\alpha(u)$, we must have a sequence of points (λ_n, u_n) along $\mathcal{C}_0$ with $\lambda_n < \lambda_{n+1}$, $0 < u_n < 1$ on Ω and $u_n(x) \to 1$ for $x \in \Omega$ as $n \to \infty$. Were the set $\{u_n\colon n \geqslant 1\}$ pre-compact in $C^1(\bar{\Omega})$, there would be a subsequence (which we could relabel if need to be) so that $u_n \to 1$ in $C^1(\bar{\Omega})$. Since

$$\alpha(u_n)\nabla u_n \cdot \eta + \big(1-\alpha(u_n)\big)u_n = 0$$

on $\partial\Omega$, we obtain by passing to the limit that

$$1 - \alpha(1) = 0,$$

so that $\alpha(1) = 1$. So $\mathcal{D}$ can be bounded in $C^{1,\gamma}(\bar{\Omega})$ only if $\alpha(1) = 1$.

As previously noted, (4.35) boils down to having $\pi(\mathcal{C}_0) = \{\lambda \in \mathbb{R}\colon (\lambda, u) \in \mathcal{C}_0$ for some $u \in \mathcal{D}\}$ be bounded. Now for any sequence (λ_n, u_n) along $\mathcal{C}_0$ with $\lambda_n \to +\infty$ as $n \to \infty$, it is the case that $u_n(x) \to 1$ for any $x \in \Omega$. Since $\alpha(1) = 1$, there is no obvious reason why such a sequence $\{u_n\}_{n\geqslant 1}$ should not converge to 1 in $C^1(\bar{\Omega})$. So it is at least plausible that one could have $\mathcal{D}$ bounded in $C^{1,\gamma}(\bar{\Omega})$ without $\pi(\mathcal{C}_0)$ being bounded in $\mathbb{R}$. On the other hand, it may be possible to rule out an unbounded $\pi(\mathcal{C}_0)$ if certain a priori estimates hold that are uniform in λ. Deciding whether $\pi(\mathcal{C}_0)$ can be unbounded in $\mathbb{R}$, and if so, under what additional conditions on $\alpha(u)$, if any, are worthy topics for further investigation in our estimation, and we plan to explore these issues further in subsequent work.

Let us return briefly to the discussion at the end of the previous section. In the case where $\lambda^1_{\alpha(0)}(\Omega) < \lambda_0$ for some given choice of α satisfying the conditions of Theorem 4.1 with $\alpha(0) > 0$, the equilibrium solution $(\lambda, 1)$ of (1.1) is linearly unstable for λ in the neighborhood $(0, \lambda_0)$ of $\lambda^1_{\alpha(0)}(\Omega)$. Consequently, as noted in Section 3, obtaining an Allee effect in the system on the basis that both $(\lambda, 0)$ and $(\lambda, 1)$ are stable equilibrium solutions of (1.1) for an interval of λ values is ruled out. However, as observed in Section 2, in this instance, we can still detect an Allee effect in the system for values of λ in the interval $(\lambda^1_{\alpha(0)}(\Omega) - \delta, \lambda^1_{\alpha(0)}(\Omega))$ when the conditions (2.23) and (2.29) hold on the basis of the instability of the positive equilibria (λ, u) which emanate from $(\lambda^1_{\alpha(0)}(\Omega), 0)$, as in [4]. Such equilibria are the elements of $\mathcal{C}_0$ which lie in a small enough neighborhood of $(\lambda^1_{\alpha(0)}(\Omega), 0)$ in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$. In such a case, $\pi(\mathcal{C}_0)$ has a positive minimum value $\lambda_*(\mathcal{C}_0) < \lambda^1_{\alpha(0)}(\Omega)$, and for all $\lambda \in (\lambda_*(\mathcal{C}_0), \lambda^1_{\alpha(0)}(\Omega))$ there are at least two equilibrium solutions (λ, u) of (1.1) with u taking values in $(0, 1)$ on Ω so that $(\lambda, u) \in \mathcal{C}_0$. For λ close enough to $\lambda^1_{\alpha(0)}(\Omega)$, one of these equilibrium solutions, say u_1, is the minimal positive equilibrium solution to (1.1) and is linearly unstable as a solution to (1.1). If there are only two elements of $\mathcal{C}_0$ for this value of λ, say $\{u_1, u_2\}$, then $u_2 > u_1$ on $\bar{\Omega}$ and all solutions to (1.1) with initial data between u_1 and u_2 converge to u_2 as $t \to \infty$. □

5. A spectral theoretic observation

Recall that in Section 3 the question of bifurcation of equilibria to (1.1) from the ray of equilibria $(\lambda, 1)$, $\lambda \in \mathbb{R}$, when $\alpha(1) = 1$ and $\alpha'(1) > 0$ led to the eigenvalue problem

$$\begin{aligned} \nabla^2\phi - \lambda\phi &= 0 && \text{in } \Omega, \\ \nabla\phi \cdot \vec{\eta} - \alpha'(1)\phi &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

Since $\alpha'(1)$ is assumed to be positive, the Robin boundary condition in (5.1) is such that standard elliptic theory as in [9] was not directly applicable for our purposes. We circumvented this issue via the change of variables (3.4), which enabled us to recast the equilibria to (1.1) as solutions to an equivalent boundary value problem on Ω in which the boundary condition was of a form to which standard elliptic theory applied. In the reformulated problem, the linear eigenvalue problem corresponding to (5.1) is (3.11). Both problems admit nontrivial solutions for precisely the same values of the parameter λ. As a result, we know from our analysis in Section 3 that (5.1) has nontrivial solutions for a discrete set $\{\lambda_n\}_{n=0}^{\infty}$ of values with $\lambda_0 > 0$, $\lambda_0 > \lambda_1$, $\lambda_n \geqslant \lambda_{n+1}$ and $\lim_{n\to\infty} \lambda_n = -\infty$.

The equilibrium solutions (λ, u) to (1.1) of applied interest in our context have the property that $0 < u < 1$ and that $\lambda \geqslant 0$. We observe in Section 3 that such solutions can emanate from the ray of solutions $(\lambda, 1)$ only at $\lambda = \lambda_0$, since λ_0 is the only element in the collection $\{\lambda_n\}_{n \geqslant 0}$ for which the corresponding ϕ in (5.1) is of one sign on Ω. Solutions emanating from $(\lambda, 1)$ at any other value λ_n will still be positive on Ω, but must take on values both above and below 1, since the eigenfunctions corresponding to λ_n in (5.1) must change sign on Ω.

As noted, we know from our analysis in Section 3 that $\lambda_0 > 0$. Indeed, we know from (3.21) and (3.26) that $\lim_{\alpha'(1)\to 0} \lambda_0 = 0$ and $\lim_{\alpha'(1)\to +\infty} \lambda_0 = +\infty$, so that λ_0 may assume any positive value. Whether any other value λ_n may be nonnegative is an interesting question. The purpose of this section is to answer this question when $\Omega = (a, b)$, a bounded interval in $\mathbb{R}^1$. To this end consider

$$\begin{aligned} &\phi'' - \lambda\phi = 0 \quad \text{on } (a, b), \\ &-\phi'(a) - \alpha'(1)\phi(a) = 0, \\ &\phi'(b) - \alpha'(1)\phi(b) = 0. \end{aligned} \tag{5.2}$$

For $x \in (0, 1)$, define $\psi(x)$ by

$$\psi(x) = \phi\big(a + (b - a)x\big). \tag{5.3}$$

It follows from (5.2) and (5.3) that ψ satisfies

$$\begin{aligned} &\psi'' - \lambda(b - a)^2\psi = 0 \quad \text{on } (0, 1), \\ &-\psi'(0) - \alpha'(1)(b - a)\psi(0) = 0, \\ &\psi'(1) - \alpha'(1)(b - a)\psi(1) = 0. \end{aligned} \tag{5.4}$$

So from (5.4) we see that to determine if any of $\{\lambda_n\}_{n\geqslant 1}$ may be nonnegative in (5.2), we can assume without loss of generality that $(a, b) = (0, 1)$ in (5.2).

So now assume that $(a, b) = (0, 1)$ in (5.2). One may readily check that the boundary conditions in (5.2) are symmetric in the sense of [16, Section 5.3] so that all eigenvalues are real valued. If one looks for an eigenvalue $\lambda = \mu^2$ of (5.2) with $\mu > 0$, then ϕ must be of the form

$$\phi(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

on $(0, 1)$. Plugging into the boundary conditions, an easy computation shows that such a λ is an eigenvalue of (5.2) provided

$$\tanh\mu = \frac{2\alpha'(1)\mu}{\mu^2 + [\alpha'(1)]^2}. \tag{5.5}$$

Now $\tanh\mu$ is a strictly increasing concave down function on $[0, \infty)$ with the value 0 at $\mu = 0$ and the limit $\lim_{\mu\to\infty} \tanh\mu = 1$. If we set $f(\mu) = \frac{2\alpha'(1)\mu}{\mu^2+[\alpha'(1)]^2}$, then

$$f'(\mu) = \frac{2\alpha'(1)[[\alpha'(1)]^2 - \mu^2]}{[\mu^2 + [\alpha'(1)]^2]^2}$$

and

$$f''(\mu) = \frac{4\alpha'(1)\mu[\mu^2 - 3[\alpha'(1)]^2]}{[\mu^2 + [\alpha'(1)]^2]^3}.$$

Hence $f(\mu)$ is increasing on $[0, \alpha'(1))$ with the value 0 at $\mu = 0$ and decreasing on $(\alpha'(1), \infty)$ with $\lim_{\mu\to\infty} f(\mu) = 0$. It is concave downward on $(0, \sqrt{3}\alpha'(1))$ and concave upward on $(\sqrt{3}\alpha'(1), \infty)$. The maximum value of $f(\mu)$ is 1 occurring at $\mu = \alpha'(1)$. The graphs are as in Fig. 1. As a result, there can be either one or two positive values of $\mu > 0$ for which (5.5) holds and hence by extension either one or two positive eigenvalues of (5.2). The determining

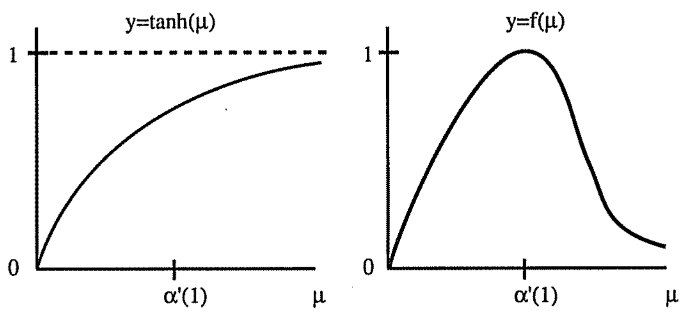

Fig. 1. Sketches of the graphs of $y = \tanh(\mu)$ and $y = f(\mu)$ from (5.5).

factor in this regard is the value of $f'(0)$. Since $\frac{d}{d\mu}(\tanh\mu) = \operatorname{sech}^2(\mu)$, $\frac{d}{d\mu}(\tanh\mu)|_{\mu=0} = 1$. If $f'(0) < 1$, the graph of $y = f(\mu)$ lies below that of $y = \tanh\mu$ for $0 < \mu \ll 1$. In this case, there must be two intersections of the curves for positive μ and hence two positive eigenvalues of (5.2). Now $f'(0) = \frac{2}{\alpha'(1)}$ so having $f'(0) < 1$ requires $\alpha'(1) > 2$. On the other hand, when $\alpha'(1) < 2$, $f'(0) > 1$ and the graph of $y = f(\mu)$ lies above the graph of $y = \tanh\mu$ for $0 < \mu \ll 1$. In this case, there is a unique positive eigenvalue of (5.2). When $\alpha'(1) = 2$, $f'(0) = 1$ and the two curves have the same tangent line at $(0, 0)$. However, one may calculate that $\lim_{\mu\to 0^+} \frac{(\tanh\mu)''}{\mu} = -2 < -\frac{3}{2} = \lim_{\mu\to 0^+} \frac{f''(\mu)}{\mu}$. Thus the graph of $y = f(\mu)$ lies above that of $y = \tanh\mu$ for $0 < \mu \ll 1$, so that there is a unique positive eigenvalue of (5.2) in this case.

To determine if 0 can be an eigenvalue of (5.2), one sets $\phi(x) = A + Bx$, so that $\phi(0) = A$, $\phi(1) = A + B$ and $\phi'(x) \equiv B$. The boundary conditions in (5.2) become

$$-B - \alpha'(1)A = 0 = B - \alpha'(1)(A + B),$$

which simplifies to $B(\alpha'(1) - 2) = 0$. Hence $\alpha'(1) = 2$ is the only value of $\alpha'(1)$ for which 0 is an eigenvalue of (5.2). Consequently, we may now conclude that the second eigenvalue λ_1 in the collection $\{\lambda_n\}_{n\geqslant 0}$ is positive when $\alpha'(1) > 2$, zero when $\alpha'(1) = 2$ and negative when $\alpha'(1) \in (0, 2)$.

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